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Random Eigenvalue Problems in Structural Dynamics: An Experimental Investigation

S. Adhikari*

Swansea University, Swansea, United Kingdom

A. Srikantha Phani†

The University of British Columbia, Vancouver, Canada

D. A. Pape‡

Central Michigan University, Mt. Pleasant, MI, USA

An experimental investigation of uncertainty in natural frequencies of linear structures *estimated* from *measured* frequency response function (FRF) under dynamic loading is presented. Experiments were conducted on one hundred nominally identical realizations of a fixed-fixed beam; each realisation was obtained by placing twelve identical masses at random spatial locations (generated by a computer) along the length of the beam. The total random mass is about 2% of the total mass of the beam. These experiments represent parametric uncertainty in the mass matrix, and hence may be useful for the validation of many random eigenvalue analysis and prediction methods currently available to structural dynamicist. Predictions from Monte Carlo simulation of deterministic finite elements model are compared with experiments. It is concluded that the method of estimation of natural frequencies from FRFs and the spatial location of the measurements has significant influence upon the first two moments (mean and standard deviation) of the natural frequency ensemble. Furthermore, whilst the Monte Carlo simulation estimates of the mean and standard deviation are in reasonable agreement with experiments at higher frequencies, the probability density function differ appreciably, within the limits of the sample size investigated in this study.

*Professor of Aerospace Engineering, School of Engineering, Swansea University, Singleton Park, Swansea SA2 8PP, UK, Email: S.Adhikari@swansea.ac.uk; AIAA Senior Member.

†Assistant Professor, Department of Mechanical Engineering, The University of British Columbia, 2054-6250 Applied Science Lane, Vancouver, BC, Canada, V6T 1Z4

‡Department of Engineering and Technology, Central Michigan University, Mt. Pleasant, MI, 48859, USA

I. Introduction

CHAracterization of natural frequencies and mode-shapes requires the solution of a linear eigenvalue problem in the analysis and design of engineering systems subjected to dynamic loads. This problem could either be a differential eigenvalue problem or a matrix eigenvalue problem, depending on whether a continuous model or a discrete model is envisaged. The description of real-life engineering structural systems is inevitably associated with some amount of uncertainty. Parametric uncertainty pertains to material and geometric properties, boundary conditions and applied loads. When we take account of these parametric uncertainties, it is imperative to solve *random eigenvalue problems* to obtain the dynamic response statistics, such as the mean and standard deviation of displacement and stress amplitudes. Random eigenvalue problems also arise in the stability analysis, or critical buckling loads calculation, of linear structural systems with random imperfections. Random eigenvalue problem arising due to parametric uncertainty can be efficiently formed using the stochastic finite element method, see for example.[1–8] The study of probabilistic characterization of the eigensolutions of random matrix and differential operators is now an important research topic in the field of stochastic structural mechanics. The paper by Boyce,[9] the book by Scheidt and Purkert[10] and the review papers by Ibrahim,[11] Benaroya,[12] Manohar and Ibrahim,[13] and Manohar and Gupta[14] are useful sources of information on early work in this area of research which also provide a systematic account of different approaches to random eigenvalue problems.

The majority of the studies reported on random eigenvalue problems are based on analytical or simulation methods. Often simulation based methods are used to validate approximate but relatively fast prediction tools (such as perturbation based methods). Experimental results are rare because of difficulties such as (a) cost involved in generating nominally identical samples of a structural system, (b) the resources and effort involved in testing a large number of samples, (c) the repetitive nature of the experimental procedure and (d) ensuring that different samples are tested in exactly the same way so that no uncertainty arises due to the measurement process. In spite of these difficulties some authors have conducted experimental investigations on random dynamical systems. Kompella and Bernhard[15] measured 57 structure-borne frequency response functions at driver microphones for different pickup trucks. Fahy[16] (page 275) reported measurements of FRFs on 41 nominally identical beer cans. Both of these experiments show variability in nominally identical engineered systems. Friswell et al.[17] reported two experiments where random systems were 'created' in the laboratory for the purpose of model validation. The first experiment used a randomly moving mass on a free-free beam and the second experiment comprised a copper pipe with uncertain internal pressure. Fifty nominally identical random samples were created and tested for both experiments.

In contrast with analytical studies, in the present experimental study, the eigenvalues are deduced from the measured frequency response functions using system identification techniques. Thus additional uncertainties may likely to be introduced by the method of system identification employed even if other uncertainties, within the control of the experimenter, were minimized. Two system identification techniques are contrasted in this regard: Rational Fractional Polynomial (RFP) method,[18] and the Nonlinear Least-Squares (NLS) technique.[18] Each system identification technique is applied to the experimental test case, described later. The difference between this data and previous experimental data is that the tests are closely controlled and the uncertainty can be considered to be 'known' for all practical purposes. This allows one to model uncertainty, propagate it through dynamical models and compare the results with experiments.

We begin with a brief introduction to random eigenvalue problems in Section II. The experiment, described in Section III is on a fixed-fixed beam with twelve masses placed at random locations. The total amount of 'random masses' is about 2% of the total mass of the beam. This experiment is aimed at simulating 'random errors' in the mass matrix. One hundred nominally identical dynamical systems are created and tested separately. The probabilistic characteristics of the frequency response function are discussed in the low, medium and high frequency ranges. The data presented here are available on the world wide web for research purposes. The web address is http://engweb.swan.ac.uk/∼adhikaris/uq/. This data may be useful to validate different uncertainty quantification and propagation methods in structural dynamics.

II. Random Eigenvalue Problems

The random eigenvalue problem of undamped or proportionally damped discrete, or discretised continuous, systems can be expressed by

$$\mathbf{K}\boldsymbol{\phi}_j = \lambda_j \mathbf{M}\boldsymbol{\phi}_j \tag{1}$$

where λ_j and $\boldsymbol{\phi}_j$ are the eigenvalues (natural frequency squared) and the eigenvectors (mode shapes) of the dynamical system. It is assumed that $\mathbf{M}$ and $\mathbf{K}$ are symmetric and positive definite random matrices so that all the eigenvalues are real and positive. We consider randomness of the system matrices of the following form

$$\mathbf{M} = \overline{\mathbf{M}} + \delta\mathbf{M} \quad \text{and} \quad \mathbf{K} = \overline{\mathbf{K}} + \delta\mathbf{K}. \tag{2}$$

Here, $\overline{(\bullet)}$ and $\delta(\bullet)$ denotes the nominal (deterministic) and random parts of $(\bullet)$ respectively. Without any loss of generality it may be assumed that $\delta\mathbf{M}$ and $\delta\mathbf{K}$ are zero-mean random matrices. We further assume that the random parts of the system matrices are small and also they preserve the symmetry, and positive definiteness of the mass matrix, of the perturbed random system. Note that no assumptions on the type of randomness, (for example Gaussian) is assumed at this stage.

The central aim of studying random eigenvalue problems is to obtain the joint probability density function (jpdf) of the eigenvalues and the eigenvectors. The current literature on random eigenvalue problems in engineering systems is dominated by the mean-centered perturbation methods.[19–28] These methods work well when the uncertainties are small and the parameter distribution is Gaussian. Some researchers have proposed methods which are not based on mean-centered perturbation method. Grigoriu[29] examined the roots of characteristic polynomials of real symmetric random matrices using the distribution of zeros of random polynomials. Recall that eigenvalues are the roots of the characteristic polynomial. Lee and Singh[30] proposed a direct matrix product (Kronecker product) method to obtain the first two moments of the eigenvalues of discrete linear systems. Nair and Keane[31] proposed a stochastic reduced basis approximation which can be applied to discrete or discretized continuous dynamic systems.

Hála[32] and Mehlhose et al.[33] used a Ritz method to obtain closed-form expressions for moments and probability density functions of the eigenvalues (in terms of Chebyshev-Hermite polynomials). Szekely and Schueller,[34] Pradlwarter et al.[35] and Du et al.[36] considered simulation based methods to obtain eigensolution statistics of large systems. Ghosh et al.[37] used a polynomial chaos expansion for random eigenvalue problems. Adhikari[38] considered complex random eigenvalue problems associated with non-proportionally damped systems. Verhoosel et al.[39] proposed an iterative method that can be applied to non-symmetric random matrices also. Rahman[40] developed a dimensional decomposition method which does not require the calculation of eigensolution derivatives. Recently Adhikari[41,42] and Adhikari and Friswell[43] have proposed an asymptotic approach to obtain joint and higher-order statistics of the eigenvalues of randomly parametered dynamical systems.

Under special circumstances when the matrix $\mathbf{H} = \mathbf{M}^{-1}\mathbf{K} \in \mathbb{R}^{N\times N}$ is Gaussian unitary ensemble (GUE) or Gaussian orthogonal ensemble (GOE) an exact closed-form expression can be obtained for the joint pdf of the eigenvalues using random matrix theory (RMT). See the book by Mehta[44] and references therein for discussions on random matrix theory. RMT has been extended to other type of random matrices. If **H** has Wishart distribution then the exact joint pdf of the eigenvalues can be obtained from Muirhead[45] (Theorem 3.2.18). Edelman[46] obtained the pdf of the minimum eigenvalue (first natural frequency squared) of a Wishart matrix. A more general case when the matrix **H** has β-distribution has been obtained by Muirhead[45] (Theorem 3.3.4) and more recently by Dumitriu and Edelman.[47] Unfortunately the system matrices of real structures may not always follow such distributions and consequently some kind of approximate analysis is required.

In this paper two experiments are used to study the random eigenvalue analysis methods available in literature. Uncertainties introduced in these experiments are not suitable for applying the stochastic finite element method.[2] As a result we have used Monte Carlo simulation approach to generate the ensembles of $\delta\mathbf{M}$ and $\delta\mathbf{K}$ and consequently the eigenvalues.

III. Random Eigenvalues of a Fixed-Fixed Beam

A. System Model and Experimental Setup

A steel beam with uniform rectangular cross-section is used for the experiment. The details of this experiment has been described by Adhikari et al.[48] Here we give a very brief overview. The physical and geometrical properties of the steel beam are shown in table 1. A steel ruler of length

Beam Properties	Numerical values
Length (L)	1200 mm
Width (b)	40.06 mm
Thickness (t_h)	2.05 mm
Mass density (ρ)	7800 Kg/m^3
Young's modulus (E)	2.0×10^5 MPa
Total weight	0.7687 Kg

Table 1. Material and geometric properties of the beam considered for the experiment

1.5m is used for the ease of placing masses at predetermined locations. These locations were generated by a random number generator. The ruler is clamped between 0.05m and 1.25m so that the effective length of the vibrating beam is 1.2m. The overall experiential setup is shown in Figure 1. The end clamps are screwed into two heavy steel blocks which in turn are fixed to a table with bolts.

Twelve equal attachable magnetic masses are used to simulate a randomly varying mass distribution. The magnets are cylindrical in shape and 12.0 mm in length and 6.0 mm in diameter. Some

Figure 1. The test rig for the fixed-fixed beam.

of the attached masses for a sample realization are shown in Figure 2. Each of them weights 2g

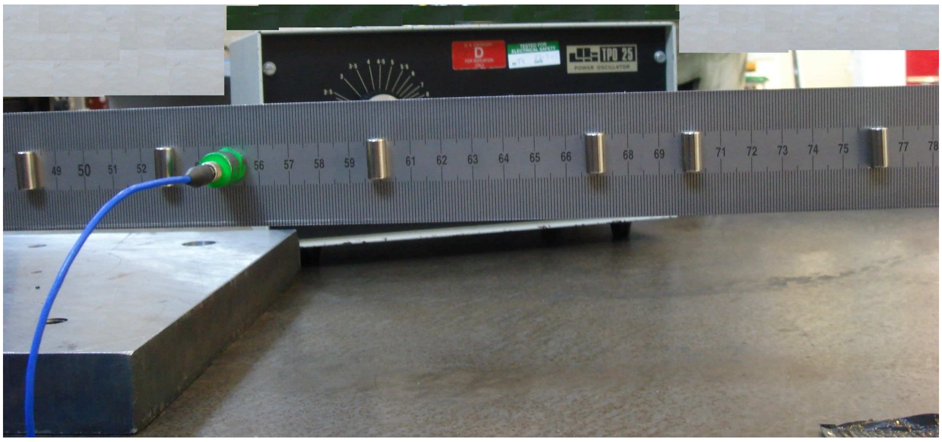


Figure 2. Attached masses (magnets) at random locations. In total 12 masses, each weighting 2g, are used.

so that the total amount of variable mass is 1.6% of the mass of the beam. The location of the 12 masses are assumed to be between 0.2m and 1.0m of the beam. A uniform distribution with 100 samples is used to generate the mass locations. The equation of motion of the 'mass loaded beam' can be expressed as

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + m\,\ddot{w}(x,t) + \sum_{j=1}^{12} m_r\,\ddot{w}(x_{r_j},t) + \sum_{j=1}^{3} m_a\,\ddot{w}(x_{a_j},t) + m_b\,\ddot{w}(x_b,t) = f(x,t). \quad (3)$$

where EI is the bending stiffness of the beam, x is the spatial coordinate along the length of the beam, t is the time, $w(x,t)$ is the time dependent transverse deflection of the beam, $f(x,t)$ is the applied time depended load on the beam, m is the mass per unit length of the beam and L is the length of the beam. For the random system, an in-house finite element code was developed to implement the discretized version of equation (3).

B. Experimental Methodology

A 32 channel LMS^{TM} system is used to conduct the experiment. Three main components of the implemented experiment are (a) excitation of the structure, (b) sensing of the response, and (c) data acquisition and processing. In this experiment we used a shaker to act as an impulse hammer. The problem with using the usual manual hammer is that it is in general difficult to hit the beam exactly at the same point with the same amount of force for every sample run. The shaker generates impulses at a pulse rate of 20s and a pulse width of 0.01s. Using the shaker in this way we have tried to eliminate any uncertainties arising from the input forces. This innovative experimental technique is designed to ensure that the resulting uncertainty in the response arises purely due to the random locations of the attached masses. Figure 3 shows the arrangement of the shaker. We

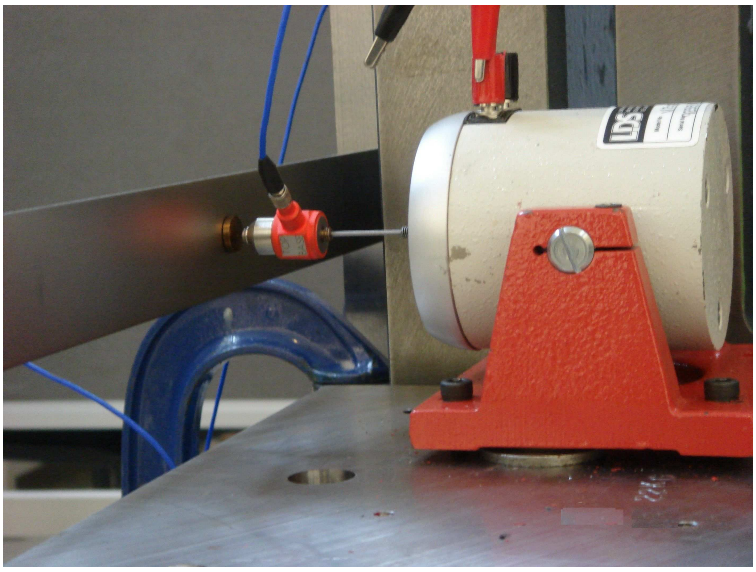


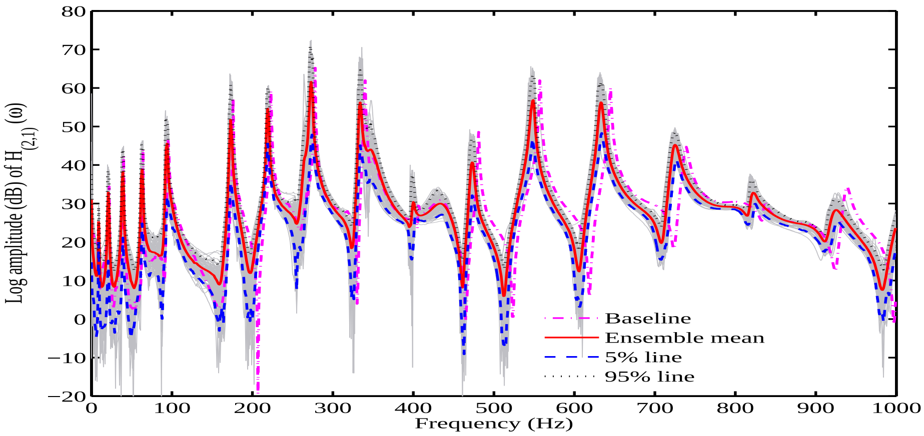
Figure 3. The shaker used as an impulse hammer using Simulink^{TM}. A hard steel tip used.

have used a small circular brass plate weighting 2g to take the impact from the shaker. This is done in order to obtain the driving point frequency response function. In this experiment three accelerometers are used as the response sensors. The location of the sensors are 23 cm (Point1), 50 cm (Point2, also the actuation point) and 102 cm (Point3) from the left end of the beam. These locations are selected such that two of them are near the two ends of the beam and one is near the middle of the beam. The exact locations are calculated such that the nodal lines of the first few bending modes can be avoided. The steel tip used in the experiment gives clean data up to approximately 4500 Hz. Here we consider modes upto 1kHz only.

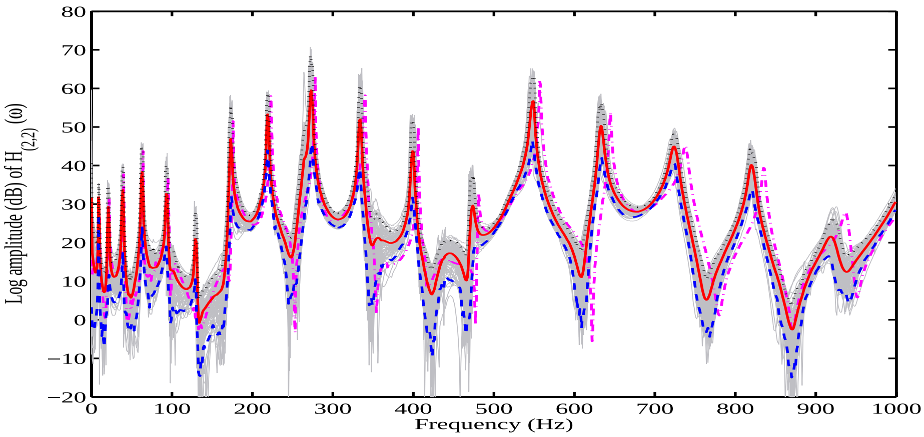
Figure 4 shows the amplitude of the frequency response function (FRF) at points 1, 2 and 3 of the beam without any masses (the baseline model). In the same figure 100 samples of the amplitude of the FRF are shown together with the ensemble mean, 5% and 95% probability lines. The ensemble mean follows the result of the baseline system closely only in the low frequency range. The relative variance of the amplitude of the FRF remains more or less constant.

C. Eigenvalue Statistics

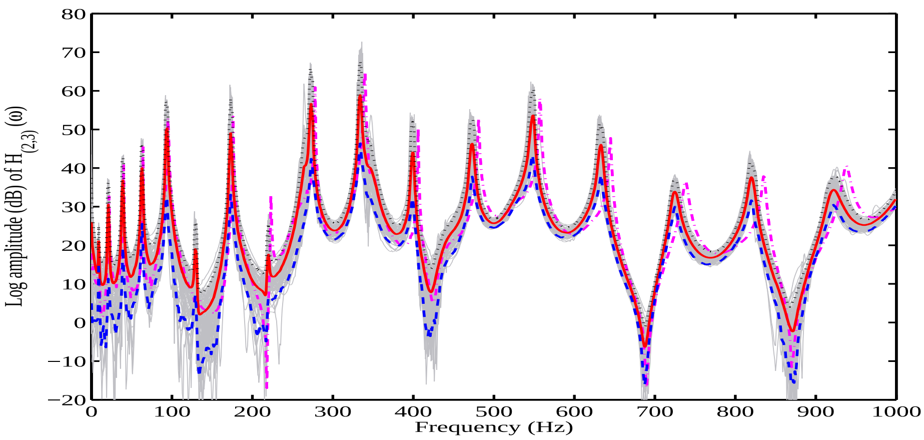
There are numerous techniques available for extracting modal parameters from FRF data. In general, these methods may be classified as single degree of freedom (SDOF) methods or multiple



(a) Point 1 (23 cm from the left end)



(b) point 2 (the driving point FRF, 50 cm from the left end)

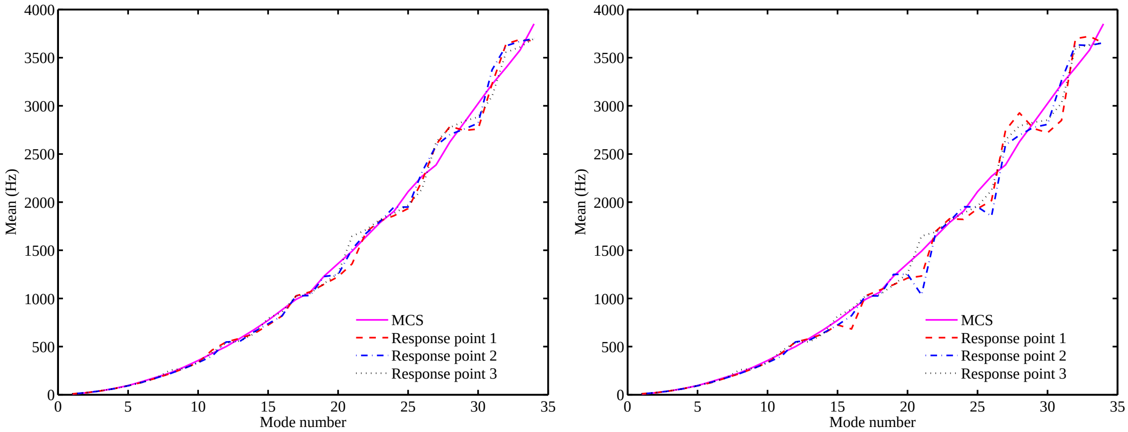


(c) point 3 (102 cm from the left end)

Figure 4. Measured amplitudes of the FRF of the beam with 12 randomly placed masses. 100 FRFs, together with the ensemble mean, 5% and 95% probability points are shown.

degree of freedom (MDOF) methods, depending upon whether one chooses to fit a curve to a single mode or to multiple modes. Because of the large number of modes observed in the frequency range of interest, selection of a MDOF technique is preferred. In the present work the

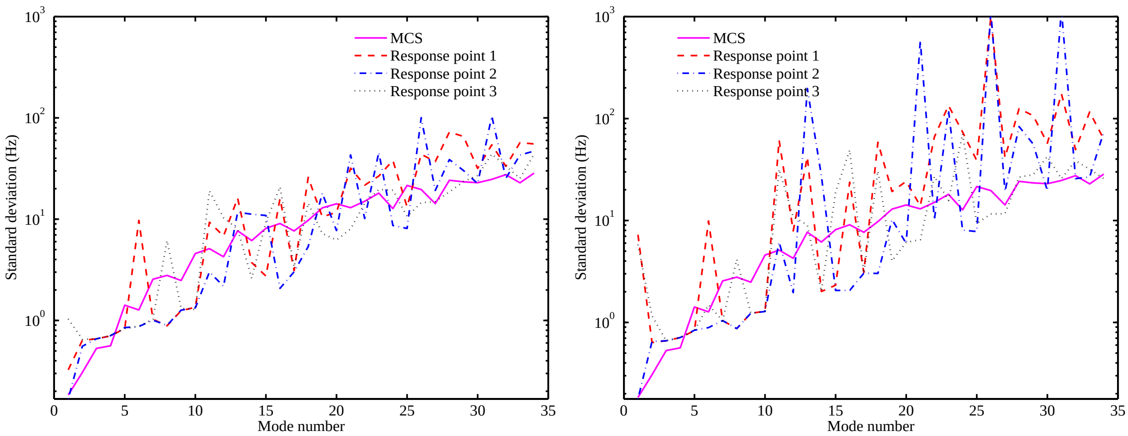
Rational Fraction Polynomial (RFP) method and a Nonlinear Least-Squares technique[18] are used. The detailed statistical analysis of the natural frequencies and comparison with analytical method is shown in Figure 5 to Figure 8. The numerical values behind these plots are given in the appendix



(a) Rational fraction polynomial method (b) Nonlinear least-square method

Figure 5. Comparison of the mean of the natural frequencies of the fixed-fixed beam obtained using the direct Monte Carlo Simulation and experimental results extracted using two different methods.

(table 2) for the purpose of possible comparisons using other analytical methods not considered in our paper. In these figures the indicated system identification method was used to extract the natural frequencies within 2 kHz from the frequency responses measured at three different spatial points on the beam. The ensemble statistics for each spatial location and for each identification method can be compared via Figure 5 and Figure 6. In Figure 5 the ensemble mean calculated from



(a) Rational fraction polynomial method (b) Nonlinear least-square method

Figure 6. Comparison of the standard deviation of the natural frequencies of the fixed-fixed beam obtained using the direct Monte Carlo Simulation and experimental results extracted using two different methods.

the *identified* natural frequencies from *measured* frequency response functions, are compared with the Monte Carlo simulations. It is instructive to compare the degree of agreement obtained by the two methods and the variation in mean with spatial location of the sensor or response measurement point. Significant differences in the mean for the three response points can be observed in the high frequency regime. It can also be seen that the *global* nonlinear least squares method exhibits significant variability. This is also reflected in Figure 6.

The normalized probability density function (pdf) plots shown in Figure 7 and Figure 8 compare the Monte Carlo simulation results with experimental data. Suppose ω_j is the random variable

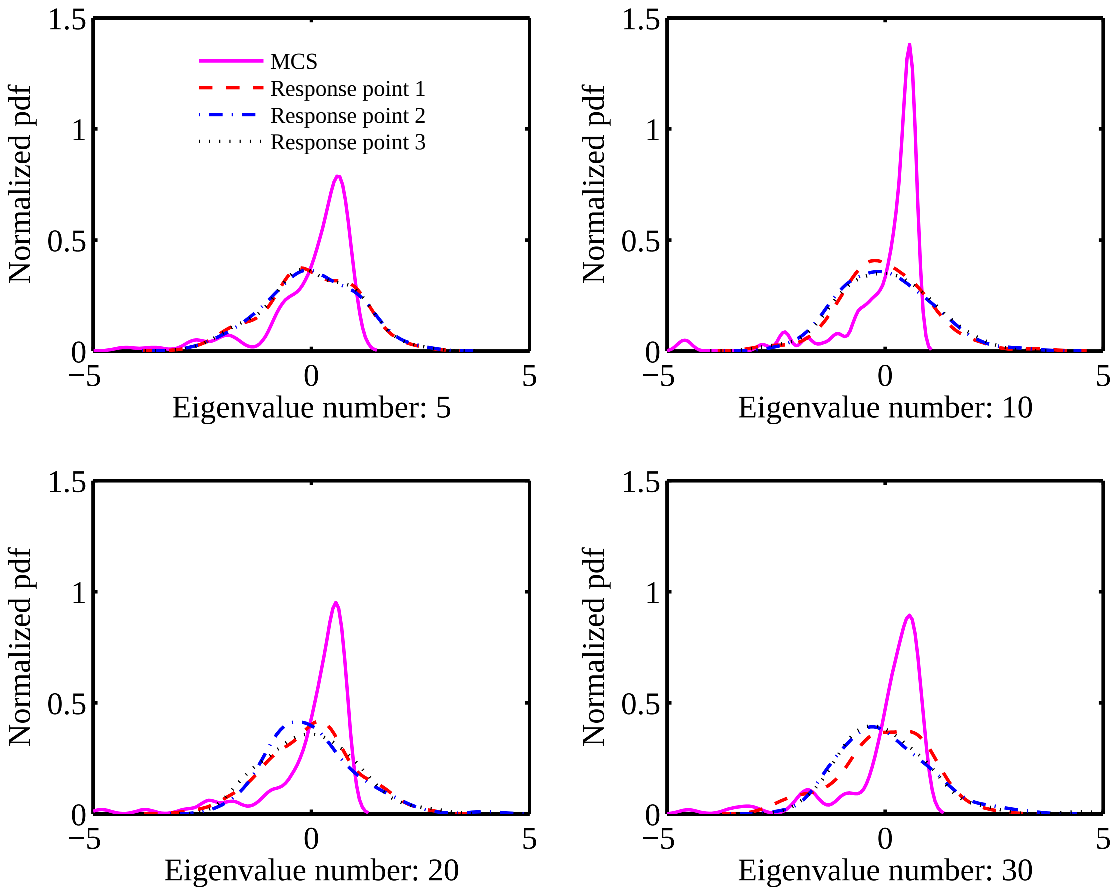


Figure 7. Comparison of the probability density functions of four selected natural frequencies of the fixed-fixed beam obtained using the direct Monte Carlo Simulation and experimental results extracted using the rational fraction polynomial method.

describing the jth natural frequency with mean $\overline{\omega}_j$ and standard deviation σ_{ω_j}. In these plots we have normalized the natural frequencies as

$$\widetilde{\omega}_j = \frac{\omega_j - \overline{\omega}_j}{\sigma_{\omega_j}} \tag{4}$$

The increasing variability as one proceeds higher in the mode sequence is obvious. The agreement at the level of pdfs is far from satisfactory. This suggests that even though the first two moments can be predicted with reasonable accuracy using Monte Carlo simulations the higher moments may not agree. While experimentally measured normalized pdfs are closer to Gaussian ensemble, albeit with non-zero mean, the Monte Carlo simulations are not.

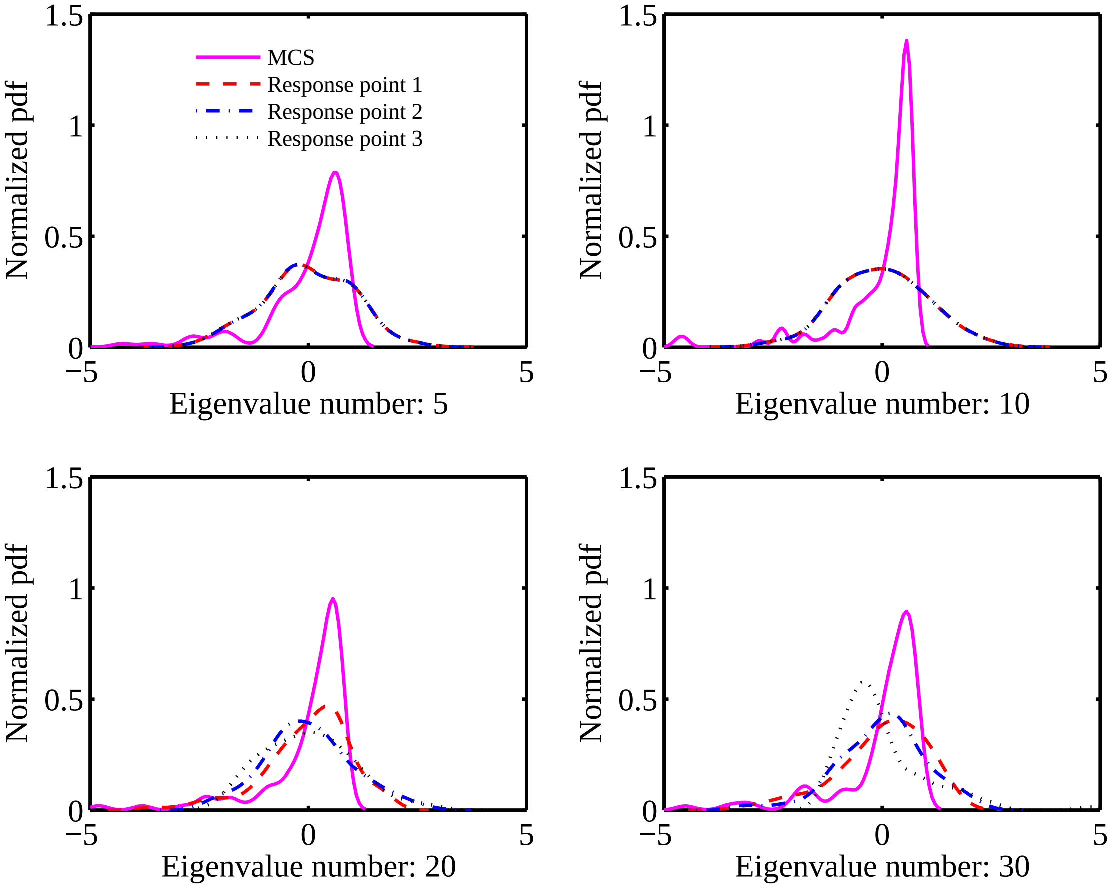


Figure 8. Comparison of the probability density functions of four selected natural frequencies of the fixed-fixed beam obtained using the direct Monte Carlo Simulation and experimental results extracted using the nonlinear least-square method.

IV. Conclusions

The statistics of the eigenvalues of discrete linear dynamic systems with uncertainties have been considered using experimental methods. One hundred nominally identical beams are created and individually tested using experimental modal analysis. Special measures have been taken so that the uncertainty in the natural frequencies only arises from the randomness in the mass and oscillator locations and the experiments are repeatable with minimum changes. Such novel measures include: (a) the use of a shaker as an impact hammer to ensure a consistent force and location for all of the tests, (b) the use of a ruler to minimize the error in measuring the mass locations in the beam experiment, (d) the use of magnets as attached masses for the ease of placement in the beam experiment.

Two methods, namely the rational fraction polynomial method and a nonlinear least-squares technique are used to extract the eigenvalues. These methods are applied to three FRFs for the beam experiment. This implies that each of the two methods was applied to 300 FRFs for the beam experiment. The following conclusions emerge from this study:

- The ensemble statistics such as mean and standard deviation for natural frequencies vary with the spatial location of the measured FRFs and the type of the system identification technique chosen to estimate the natural frequencies.

- Whilst a reasonable predictions for the mean and the standard deviations may be obtained using the Monte Carlo Simulation, higher moments, and hence the pdfs can be significantly different.
- In some cases, the differences in pdfs arising from different points and different identification methods can be more than those obtained from the Monte Carlo Simulation.

It should be recalled that the above conclusions are based on a sample size of 100. Nevertheless, these results perhaps highlight the need for new outlook when one considerers experimental works on random eigenvalue problems.

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A. Mean and Standard Deviation of Experimentally Identified Natural Frequencies

Here we give the mean and standard deviation of all the experimentally identified natural frequencies for all the measured FRFs using the two methods. In table 2 the results for the beam experiment were shown.

Mode	Rational Fraction Polynomial			Nonlinear Least Squares		
	FRF 1	FRF 2	FRF 3	FRF 1	FRF 2	FRF 3
1	9.08± 0.33	9.24± 0.17	8.39± 1.03	10.07± 7.25	9.26± 0.18	6.73± 5.79
2	21.03± 0.64	20.89± 0.56	21.01± 0.65	21.04± 0.63	20.92± 0.65	21.08± 1.16
3	38.94± 0.66	38.92± 0.66	38.93± 0.66	38.94± 0.66	38.92± 0.66	38.93± 0.66
4	62.83± 0.70	62.81± 0.71	62.80± 0.71	62.80± 0.71	62.79± 0.71	62.79± 0.71
5	93.79± 0.84	93.47± 0.85	93.65± 0.83	93.68± 0.83	93.58± 0.84	93.64± 0.83
6	128.37± 9.71	129.56± 0.87	129.50± 0.87	125.96± 9.89	129.80± 0.89	129.65± 1.49
7	173.12± 1.03	173.50± 1.01	173.09± 1.03	173.07± 1.04	173.10± 1.04	173.06± 1.04
8	219.27± 0.89	219.08± 0.90	254.58± 6.15	219.15± 0.87	219.13± 0.87	258.00± 4.14
9	273.25± 1.25	273.27± 1.26	273.34± 1.32	273.21± 1.23	273.21± 1.23	273.20± 1.23
10	333.73± 1.35	333.93± 1.33	333.59± 1.28	333.70± 1.28	333.66± 1.28	333.68± 1.28
11	463.00± 9.33	400.79± 3.01	437.50± 19.02	433.38± 61.23	399.88± 6.15	453.62± 31.80
12	547.08± 6.82	547.82± 2.17	536.75± 10.36	547.19± 7.91	548.06± 1.96	546.44± 10.87
13	586.51± 16.17	556.38± 11.69	558.63± 7.99	582.26± 42.03	568.49±197.38	549.63± 9.04
14	633.64± 3.73	653.19± 11.19	632.86± 2.61	633.24± 2.01	642.32± 27.31	633.09± 2.02
15	723.71± 2.75	736.54± 10.88	793.22± 9.38	725.60± 2.32	723.90± 2.06	816.57± 19.01
16	818.82± 16.02	819.59± 2.08	858.80± 20.83	683.02± 23.78	820.44± 2.06	890.44± 49.07
17	1027.30± 3.11	1026.88± 3.02	1024.12± 3.38	1026.92± 3.09	1026.66± 3.04	1026.59± 3.05
18	1067.56± 25.85	1030.01± 5.37	1053.84± 14.38	1086.05± 58.56	1026.59± 3.02	1037.18± 30.62
19	1147.37± 10.53	1227.66± 18.04	1153.54± 7.22	1142.98± 19.27	1249.20± 10.22	1140.21± 4.01
20	1226.03± 11.72	1247.07± 7.73	1262.04± 6.23	1211.84± 24.22	1251.38± 5.97	1259.83± 6.10
21	1359.68± 31.90	1511.76± 43.04	1645.14± 8.01	1232.35± 13.91	1036.05±579.58	1648.12± 6.48
22	1688.71± 21.69	1676.45± 10.16	1713.01± 14.35	1695.06± 65.95	1673.54± 10.14	1689.50± 27.14
23	1805.43± 26.70	1799.71± 45.71	1818.78± 19.62	1826.81±134.14	1800.18±120.84	1807.81± 15.71
24	1858.83± 38.02	1948.64± 8.67	1884.30± 19.30	1820.68± 73.88	1952.01± 8.05	1886.24± 70.76
25	1933.49± 13.20	1949.28± 8.10	1967.87± 10.65	1932.16± 38.74	1951.21± 7.78	1957.80± 9.30
26	2220.78± 43.82	2315.33±100.49	2136.00± 14.54	2015.97±998.14	1854.72±1114.61	2122.31± 11.61
27	2597.18± 37.01	2585.93± 19.24	2623.51± 14.99	2745.13± 41.36	2592.98± 19.24	2647.11± 11.64
28	2787.48± 72.53	2703.42± 38.82	2774.62± 18.57	2926.50±125.09	2695.32± 83.54	2790.94± 26.63
29	2744.54± 65.69	2758.39± 30.22	2841.25± 23.53	2765.70±106.87	2776.99± 56.24	2828.18± 28.11
30	2759.01± 31.72	2819.70± 22.43	2883.49± 27.00	2719.83± 57.31	2806.86± 19.61	2852.38± 42.42
31	3227.53± 54.19	3373.65±104.31	3099.94± 43.95	2849.03±175.06	3263.22±1163.50	3028.99± 25.66
32	3643.10± 33.30	3623.34± 25.36	3554.33± 33.42	3697.44± 49.57	3633.21± 25.76	3598.47± 38.64
33	3689.33± 56.99	3669.52± 43.03	3611.74± 25.31	3722.14±116.06	3626.82± 26.75	3634.27± 31.89
34	3672.02± 55.41	3698.77± 47.02	3698.47± 43.93	3651.91± 62.85	3655.25± 73.61	3652.49± 25.76

Table 2. Mean and standard deviation of identified natural frequencies of the fixed-fixed beam (in Hz) from the measured FRFs at three spatial locations using two identification methods.