Doubly Spectral Stochastic Finite Element Method (DSSFEM) for Random Field Problems

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Abstract: Uncertainties in complex dynamical systems play an important role in the prediction of dynamic response in the mid and high frequency ranges. For distributed parameter systems, parametric uncertainties can be represented by random fields leading to stochastic partial differential equations. Over the past two decades spectral stochastic finite element method has been developed to discretise the random fields and solve such problems. On the other hand, for deterministic distributed parameter linear dynamical systems, spectral finite element method has been developed to efficiently solve the problem in the frequency domain. In spite of the fact that both approaches use spectral decomposition (one for the random fields and while the other for the dynamic displacement fields), there has been very little overlap between them in literature. In this paper these two spectral techniques have been unified with the aim that the unified approach would outperform any of the spectral methods considered on its own. Considering exponential and triangular autocorrelation functions for the random fields, frequency depended element stiffness, mass and damping matrices are derived for axial and bending vibration of rods. Closed-form exact expressions are derived using Karhunen-Loève expansion. Numerical examples are given to illustrate the unified spectral approach.

Key words: random field, spectral method, stochastic finite element, frequency response, Karhunen-Loève expansion

NOMENCLATURE

$\mathbf{D}$	dynamic stiffness matrix
$\mathbf{f}$	forcing function of the discretized system
$\mathbf{\Gamma}(\omega)$	a constant matrix for the shape functions
$\mathbf{N}(\mathbf{r},\omega)$	shape function vector
$\mathbf{r}$	spatial coordinate vector
$\mathbf{s}(\mathbf{r},\omega)$	a vector of elementary functions for the shape functions
$\mathbf{u}$	response vector of the discretized system
$\mathcal{D}$	space of the random field H
λ_j	jth eigenvalue corresponding to the auto-covariance kernel
Ω	sample space
$\rho(\mathbf{r},\theta)$	random mass density
θ	elements of the sample space Ω
$\varphi_j(\mathbf{r})$	jth eigenfunction corresponding to the auto-covariance kernel
$\widehat{\mathbf{u}}_e(\omega)$	nodal displacement vector for an element
$\xi_j(\theta)$	uncorrelated Gaussian random variables
$c(\mathbf{r},\theta)$	random distributed damping parameter
$C_H(\mathbf{r}_1,\mathbf{r}_2)$	covariance function of the random field H
$H(\mathbf{r},\theta)$	a random field in $\mathbf{r}$
$k_s(\mathbf{r},\theta)$	random distributed stiffness parameter
$L_1(\bullet)$	damping operator
$L_2(\bullet)$	stiffness operator
m	order of the governing differential equation
n	dimension of the element matrices
$U(\mathbf{r},t)$	spatial displacement variable
$u_e(\mathbf{r},\omega)$	displacement variable within an element
$(\bullet)_0$	deterministic part corresponding to $(\bullet)$
$(\bullet)^T$	matrix transposition
$\mathbb{C}$	space of complex numbers
$\mathbb{C}^{n\times m}$	space $n \times m$ complex matrices

1. INTRODUCTION

Spectral methods are widely used in various branches of science and engineering. Due to their general nature, the meaning of spectral methods can be very different depending on the applications and the disciplines. These differences mainly arise due to the lack of communication between different disciplines. In spite of these differences, the unifying

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factor between the spectral methods in different disciplines is that generally they are very powerful tools for the analytical and experimental treatments of wide ranging physical problems. In the context of stochastic finite element method [1–14], spectral methods have been used extensively to analytically represent the random fields describing parametric uncertainties of physical systems. In the context of structural dynamics, spectral methods have been used in random vibration problems [15–18] and for the discretisation of displacement fields in the frequency domain [19–24]. Several application of such method has been discussed in details in ref. [25]. In spite of the fact that both approaches use spectral decomposition (one for the random fields and while the other for the dynamic displacement fields), there has been very little overlap between them in literature. In this paper these two spectral techniques have been unified with the aim that the unified approach would perform better than any of the spectral methods considered on its own.

Numerical computer codes implementing physics based models are the backbone of today's dynamic analysis of complex systems. Laboratory based controlled tests are often performed to gain insight into some specific physics of a problem. Such tests can indeed lead to new physical laws improving a part of the overall science model. Test data can also be used to calibrate a known model. However, neither of these activities are enough to produce a credible numerical tool because of the several types of uncertainties which exist in the whole process of physics based computational predictions. Such uncertainties include, but not limited to (a) parametric uncertainty (e.g, uncertainty in geometric parameters, friction coefficient, strength of the materials involved); (b) model inadequacy (arising from the lack of scientific knowledge about the model which is a-priori unknown); (c) experimental error (uncertain and unknown error percolate into the model when they are calibrated against experimental results); (d) computational uncertainty (e.g, machine precession, error tolerance and the so called 'h' and 'p' refinements in finite element analysis) and (e) model uncertainty (genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos). These uncertainties must be assessed and managed for credible computational predictions.

In this paper we focus our attention to dynamical systems with parametric uncertainties. In the parametric approach, the uncertainties associated with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficient and geometric parameters are quantified using statistical methods and propagated, for example, using the stochastic finite element method. The effect of uncertainty is significant in the higher frequency ranges. In the higher frequency ranges, as the wavelengths become smaller, very fine (static) mesh size is required to capture the dynamical behaviour. As a result, the deterministic analysis itself can pose significant computational challenges. One way to address this problem is to use a spectral approach in the frequency domain [25]. The main idea here is that the displacements within an element is expressed in terms of frequency dependent shape functions. The shape functions adapt themselves with increasing frequency and consequently displacements can be obtained accurately without fine remeshing. The spectral approach have the potential to be an efficient method for mid and high frequency vibration problems provided the random fields describing parametric uncertainties can be taken into account efficiently. Here the spectral decomposition of the random files are used in conjunction with the spectral decomposition of the displacements field. It is expected that simultaneous use of these two types of spectral decompositions will result into an efficient approach for distributed dynamical systems with parametric uncertainties. Spectral finite element method in the frequency domain is briefly discussed in section 2. The essential background of spectral representation of stochastic fields is given in section 3. The general derivation of the element mass, stiffness and damping matrices using the doubly spectral stochastic finite element is given in section 4. In section 5 this general theory is applied to axially vibrating rods with uncertain properties.

2. SPECTRAL FINITE ELEMENT IN THE FREQUENCY DOMAIN

Spectral method for deterministic dynamical systems have been in use for more than three decades (see for example the book by Paz [26]). This approach, or approaches very similar to this, is known by various names such as the dynamic stiffness method [27–39], spectral finite element method [19, 20, 25] and dynamic finite element method [40, 41]. Some of the notable features of the method, which can be discerned from the available literature are

1. the mass distribution of the element is treated in an exact manner in deriving the element dynamic stiffness matrix;
2. the dynamic stiffness matrix of one dimensional structural elements taking into account the effects of flexure, torsion, axial motion, shear deformation effects and damping are exactly determinable, which, in turn, enables the exact vibration analysis of skeletal structures by an inversion of the global dynamic stiffness matrix;
3. the method does not employ eigenfunction expansions and, consequently, a major step of the traditional finite element analysis, namely, the determination of natural frequencies and mode shapes, is eliminated which automatically avoids the errors due to series truncation; this makes the method attractive for situations in which a large number of modes participate in vibration;
4. since the modal expansion is not employed, *ad hoc* assumptions concerning damping matrix being proportional to mass and/or stiffness is not necessary;
5. the method is essentially a frequency domain approach suitable for steady state harmonic or stationary random excitation problems; generalization to other type of problems through the use of Laplace transforms is also available;

6. the static stiffness matrix and the consistent mass matrix appear as the first two terms in the Taylor expansion of the dynamic stiffness matrix in the frequency parameter.

3. SPECTRAL STOCHASTIC FINITE ELEMENT METHOD

Problems of structural dynamics in which the uncertainty in specifying mass and stiffness of the structure is modeled within the framework of random fields can be treated using the stochastic finite element method [1–14]. Application of the stochastic finite element method to linear structural dynamics problems typically consists of the following key steps:

1. Selection of appropriate probabilistic models for parameter uncertainties and boundary conditions
2. Replacement of the element property random fields by an equivalent set of a finite number of random variables. This step, known as the 'discretisation of random fields' is a major step in the analysis.
3. Formulation of the system equations of motion of the form $\mathbf{D}(\omega)\mathbf{u} = \mathbf{f}$ where $\mathbf{D}(\omega)$ is the random dynamic stiffness matrix $\mathbf{u}$ is the vector of random nodal displacement and $\mathbf{f}$ is the applied forces. In general $\mathbf{D}(\omega)$ is a random symmetric complex matrix.
4. Solution of the set of complex random algebraic equation to obtain the statistics of the response vectors

Suppose $H(\mathbf{r}, \theta)$ is a random field with a covariance function $C_H(\mathbf{r}_1, \mathbf{r}_2)$ defined in a space $\mathcal{D}$. Here θ denotes an element of the (random) sample space Ω so that $\theta \in \Omega$. Since the covariance function is finite, symmetric and positive definite it can be represented by a spectral decomposition. Using this spectral decomposition, the random process $H(\mathbf{r}, \theta)$ can be expressed in a generalized fourier type of series as

$$H(\mathbf{r}, \theta) = H_0(\mathbf{r}) + \sum_{j=1}^{\infty} \sqrt{\lambda_j}\xi_j(\theta)\varphi_j(\mathbf{r}) \tag{1}$$

where $\xi_j(\theta)$ are uncorrelated random variables, λ_j and $\varphi_j(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\mathcal{D}} C_H(\mathbf{r}_1, \mathbf{r}_2)\varphi_j(\mathbf{r}_1)\mathrm{d}\mathbf{r}_1 = \lambda_j\varphi_j(\mathbf{r}_2), \quad \forall\ i = 1, 2, \cdots \tag{2}$$

The spectral decomposition in equation (1) is known as the Karhunen-Loève expansion. The series in (1) can be ordered in a decreasing series so that it can be truncated after a finite number of terms with a desired accuracy. We refer the books [2, 42] and references therein for further discussions on Karhunen-Loève expansion.

In this paper one dimensional systems are considered. Moreover, Gaussian random fields with exponentially decaying autocorrelation function is considered. The autocorrelation function can be expressed as

$$C(x_1, x_2) = e^{-|x_1 - x_2|/b} \tag{3}$$

Here the constant b is known as the correlation length and it plays an important role in the description of a random field. If the correlation length is very small, then the random process becomes close to a delta-correlated process, often know as the white noise. If the correlation length is very large compared to domain under consideration, the the random process effectively becomes a random variable. The underlying random process $H(x, \theta)$ can be expanded using the Karhunen-Loève expansion [2, 42] in the interval $-a \le x \le a$ as

$$H(x, \theta) = \sum_{j=1}^{\infty} \left[\xi_j(\theta)\sqrt{\lambda_j}\varphi_j(x) + \xi_j^*(\theta)\sqrt{\lambda_j^*}\varphi_j^*(x)\right]. \tag{4}$$

Using the notation $c = 1/b$, the corresponding eigenvalues and eigenfunctions, for odd j are given by

$$\lambda_j = \frac{2c}{\omega_j^2 + c^2}; \quad \varphi_j(x) = \frac{\cos(\omega_j x)}{\sqrt{a + \dfrac{\sin(2\omega_j a)}{2\omega_j}}} \tag{5}$$

$$\tan(\omega a) = \frac{c}{\omega} \tag{6}$$

and for even j are given by

$$\lambda_j^* = \frac{2c}{\omega_j^{*2} + c^2}; \quad \varphi_j^*(x) = \frac{\sin(\omega_j^* x)}{\sqrt{a - \dfrac{\sin(2\omega_j^* a)}{2\omega_j^*}}} \tag{7}$$

$$\tan(\omega^* a) = \frac{\omega^*}{-c} \tag{8}$$

These eigenvalues and eigenfunctions will now be used to obtain the element mass, stiffness and damping matrices.

4. GENERAL DERIVATION OF DOUBLY SPECTRAL ELEMENT MATRICES

A linear damped distributed parameter dynamical system in which the displacement variable $U(\mathbf{r}, t)$, where $\mathbf{r}$ is the spatial position vector and t is time, specified in some domain $\mathcal{D}$, is governed by a linear partial differential equation

$$\rho(\mathbf{r}, \theta)\frac{\partial^2 U(\mathbf{r}, t)}{\partial t^2} + L_1\frac{\partial U(\mathbf{r}, t)}{\partial t} + L_2 U(\mathbf{r}, t) = p(\mathbf{r}, t); \quad \mathbf{r} \in \mathcal{D}, t \in [0, T] \tag{9}$$

with linear boundary-initial conditions of the form

$$M_{1j}\frac{\partial U(\mathbf{r}, t)}{\partial t} = 0; \quad M_{2j}U(\mathbf{r}, t) = 0; \quad \mathbf{r} \in \Gamma,\ t = t_0,\ \ j = 1, 2, \cdots \tag{10}$$

specified on some boundary surface Γ. In the above equation $\rho(\mathbf{r}, \theta)$ is the random mass distribution of the system, $p(\mathbf{r}, t)$ is the distributed time-varying forcing function, L_1 is the random spatial self-adjoint damping operator, L_2 is the random spatial self-adjoint stiffness operator and M_{1j} and

M_{2j} are some linear operators defined on the boundary surface Γ. When parametric uncertainties are considered, the mass density $\rho(\mathbf{r},\theta)$ as well as the damping and stiffness operators involve random processes. Here θ denotes the random nature of the function. Frequency depended random element stiffness matrices were derived by various authors using the weighted integral approach [5, 6, 8, 43], the energy operator approach [44] and a series expansion approach [45]. Suppose the underlying homogeneous system corresponding to system (9) without any forcing is given by

$$\rho_0 \frac{\partial^2 U(\mathbf{r},t)}{\partial t^2} + L_{10}\frac{\partial U(\mathbf{r},t)}{\partial t} + L_{20}U(\mathbf{r},t) = 0 \tag{11}$$

together with suitable homogeneous boundary and initial conditions. Taking the Fourier transform of equation (11) and considering zero initial conditions one has

$$-\omega^2\rho_0 u(\mathbf{r}) + \mathrm{i}\omega L_{10}\{u(\mathbf{r})\} + L_{20}\{u(\mathbf{r})\} = 0 \tag{12}$$

Like the classical finite element method, suppose that frequency-dependent displacement within an element is interpolated from the nodal displacements as

$$u_e(\mathbf{r},\omega) = \mathbf{N}^T(\mathbf{r},\omega)\widehat{\mathbf{u}}_e(\omega) \tag{13}$$

Here $\widehat{\mathbf{u}}_e(\omega) \in \mathbb{C}^n$ is the nodal displacement vector and $\mathbf{N}(\mathbf{r},\omega) \in \mathbb{C}^n$, the vector of frequency-dependent shape functions and n is the number of the nodal degrees-of-freedom. Suppose the $s_j(\mathbf{r},\omega) \in \mathbb{C}, j = 1, 2, \cdots m$ are the basis functions which exactly satisfy equation (12). Here m is the order of the ordinary differential equation (12). It can be shown that the shape function vector can be expresses as

$$\mathbf{N}(\mathbf{r},\omega) = \mathbf{\Gamma}(\omega)\mathbf{s}(\mathbf{r},\omega) \tag{14}$$

where the vector $\mathbf{s}(\mathbf{r},\omega) = \left\{s_j(\mathbf{r},\omega)\right\}^T, \forall\, j \in \mathbb{C}^m$ and the complex matrix $\mathbf{\Gamma}(\omega) \in \mathbb{C}^{nm}$ depends on the boundary conditions. The detailed derivation will be given in full paper.

Extending the weak-form of finite element approach to the complex domain, the frequency depended $n \times n$ complex random stiffness, mass and damping matrices can be obtained as

$$\mathbf{K}_e(\omega,\theta) = \int_{\mathcal{D}_e} k_s(\mathbf{r},\theta)\mathcal{L}_2\{\mathbf{N}(\mathbf{r},\omega)\}\,\mathcal{L}_2\left\{\mathbf{N}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \tag{15}$$

$$\mathbf{M}_e(\omega,\theta) = \int_{\mathcal{D}_e} \rho(\mathbf{r},\theta)\mathbf{N}(\mathbf{r},\omega)\mathbf{N}^T(\mathbf{r},\omega)\mathrm{d}\mathbf{r} \quad \text{and} \tag{16}$$

$$\mathbf{C}_e(\omega,\theta) = \int_{\mathcal{D}_e} c(\mathbf{r},\theta)\mathcal{L}_1\{\mathbf{N}(\mathbf{r},\omega)\}\,\mathcal{L}_1\left\{\mathbf{N}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \tag{17}$$

Where, $(\bullet)^T$ denotes matrix transpose, $k_s(\mathbf{r},\theta)$ is the random distributed stiffness parameter, $\mathcal{L}_2\{\bullet\}$ is the strain energy operator, $c(\mathbf{r},\theta)$ is the random distributed damping parameter and $\mathcal{L}_1\{\bullet\}$ is the energy dissipation operator. The random fields $k_s(\mathbf{r},\theta)$, $\rho(\mathbf{r},\theta)$ and $c(\mathbf{r},\theta)$ are expanded using the Karhunen-Loève expansion (1). Using finite number of terms, each of the complex element matrices can be expanded in a spectral series as

$$\mathbf{K}_e(\omega,\theta) = \mathbf{K}_{0e}(\omega) + \sum_{j=1}^{N_\mathrm{K}} \xi_{\mathrm{K_j}}(\theta)\mathbf{K}_{je}(\omega) \tag{18}$$

$$\mathbf{M}_e(\omega,\theta) = \mathbf{M}_{0e}(\omega) + \sum_{j=1}^{N_\mathrm{M}} \xi_{\mathrm{M_j}}(\theta)\mathbf{M}_{je}(\omega) \tag{19}$$

$$\text{and} \quad \mathbf{C}_e(\omega,\theta) = \mathbf{C}_{0e}(\omega) + \sum_{j=1}^{N_\mathrm{C}} \xi_{\mathrm{C_j}}(\theta)\mathbf{C}_{je}(\omega) \tag{20}$$

Here the complex deterministic symmetric matrices, for example in the case of the stiffness matrix, can be obtained as

$$\mathbf{K}_{0e}(\omega) = \int_{\mathcal{D}_e} k_{s_0}(\mathbf{r})\mathcal{L}_2\{\mathbf{N}(\mathbf{r},\omega)\}\,\mathcal{L}_2\left\{\mathbf{N}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \quad \text{and} \tag{21}$$

$$\mathbf{K}_{je}(\omega) = \sqrt{\lambda_{\mathrm{K_j}}}\int_{\mathcal{D}_e} \varphi_{\mathrm{K_j}}(\mathbf{r})\mathcal{L}_2\{\mathbf{N}(\mathbf{r},\omega)\}\,\mathcal{L}_2\left\{\mathbf{N}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \tag{22}$$

$$\forall\, j = 1, 2, \cdots, N_\mathrm{K}$$

The equivalent terms corresponding to the mass and damping matrices can be obtained in a similar manner. Substituting the shape function from equation (14), into equations (21) and (22) one obtains

$$\mathbf{K}_{0e}(\omega) = \mathbf{\Gamma}(\omega)\widetilde{\mathbf{K}}_{0e}(\omega)\mathbf{\Gamma}^T(\omega) \quad \text{and} \tag{23}$$

$$\mathbf{K}_{je}(\omega) = \sqrt{\lambda_{\mathrm{K_j}}}\mathbf{\Gamma}(\omega)\widetilde{\mathbf{K}}_{je}(\omega)\mathbf{\Gamma}^T(\omega); \quad \forall\, j = 1, 2, \cdots, N_\mathrm{K} \tag{24}$$

where

$$\widetilde{\mathbf{K}}_{0e}(\omega) = \int_{\mathcal{D}_e} k_{s_0}(\mathbf{r})\mathcal{L}_2\{\mathbf{s}(\mathbf{r},\omega)\}\,\mathcal{L}_2\left\{\mathbf{s}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \in \mathbb{C}^{mm} \quad \text{and} \tag{25}$$

$$\widetilde{\mathbf{K}}_{je}(\omega) = \int_{\mathcal{D}_e} \varphi_{\mathrm{K_j}}(\mathbf{r})\mathcal{L}_2\{\mathbf{s}(\mathbf{r},\omega)\}\,\mathcal{L}_2\left\{\mathbf{s}^T(\mathbf{r},\omega)\right\}\mathrm{d}\mathbf{r} \in \mathbb{C}^{mm} \tag{26}$$

$$\forall\, j = 1, 2, \cdots, N_\mathrm{K}$$

The expressions of the eigenfunctions given in the previous section are valid within the specific domains defined before. One needs to change the coordinate in order to use them in equation (26). Once the element stiffness, mass and damping matrices are obtained in this manner, the global matrices can be calculated by summing the element matrices with suitable coordinate transformations as in the standard finite element method.

5. DSSFEM FOR RODS IN AXIAL VIBRATION

We consider the following method of time-frequency Fourier spectral approximation for the displacement field for axial vibration in a rod.

The equation of motion of the stochastically inhomogeneous rod under axial motion is given by

$$\frac{\partial}{\partial x}\left[AE(x,\theta)\frac{\partial U}{\partial x}\right] + c_0\frac{\partial U}{\partial t} - m(x,\theta)\frac{\partial^2 U}{\partial t^2} = 0 \qquad (27)$$

Here the axial rigidity $AE(x)$ and the mass per unit length $m(x)$ is assumed to be random fields of the following form

$$AE(x,\theta) = A_0E_0\left[1 + \epsilon_{AE}H_{AE}(x,\theta)\right] \qquad (28)$$

$$m(x) = m_0(1 + \epsilon_m H_m(x,\theta)) \qquad (29)$$

Here $H_{AE}(x,\theta)$ and $H_m(x,\theta)$ are assumed to homogeneous Gaussian random fields with zero mean and exponentially decaying autocorrelation function of the form given by equation (3). The 'strength parameters' ϵ_{AE} and ϵ_m effectively quantify the amount of uncertainty in the axial rigidity and mass per unit length of the rod. The constants A_0E_0 and $m_0 = A_0\rho_0$ are respectively the mass per unit length and axial rigidity of the underline baseline model. The equation of motion of the baseline model is given by

$$A_0\rho_0\frac{\partial^2 U(x,t)}{\partial t^2} + c_0\frac{\partial U}{\partial t} - A_0E_0\frac{\partial^2 U}{\partial x^2} = 0\,, \qquad (30)$$

where ρ_0, E_0, A_0 and c_0 are the nominal value of the density, elastic stiffness, cross sectional area and damping factor within a domain $x \in [0, L]$. With spectral expansion of the axial displacement $U(x,t)$ in the frequency-wavenumber space, one has

$$U(x,t) = \left(\tilde{u}_1 e^{-ik_0x/L} + \tilde{u}_2 e^{-ik_0(1-x/L)}\right)e^{i\omega t} = u(x)e^{i\omega t} \qquad (31)$$

where $i = \sqrt{-1}$, k_0 is the non-dimensional wavenumber for the reference model in Eq. (30), which is given by

$$k_0 = \omega L\sqrt{\frac{\rho_0}{E_0}}\sqrt{1 - \frac{ic_0}{\omega\rho_0}}\,. \qquad (32)$$

By defining spectral element nodes at $x = 0, L$ and the one axial displacement degrees of freedom at each node as $u_1(x = 0)$ and $u_2(x = L)$, one obtains the generic displacement field in terms of these two degrees of freedom as

$$\begin{aligned} u(x) =& \left[\frac{e^{-ik_0x/L} - e^{-ik_0(2-x/L)}}{1 - e^{-i2k_0}} \quad \frac{-e^{-ik_0(1+x/L)} + e^{-ik_0(1-x/L)}}{1 - e^{-i2k_0}}\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \\ =& \mathbf{N}(x,\omega)\hat{\mathbf{u}}_e(\omega) \end{aligned} \qquad (33)$$

which gives the frequency dependent shape function $\mathbf{N}(x,\omega)$ as discussed earlier but for the one-dimensional rod problem as an example. After some elementary algebra, the shape function vector in equation (33) can be expressed in the form of equation (14) as

$$\begin{aligned} &\mathbf{N}(\mathbf{r},\omega) = \mathbf{\Gamma}(\omega)\mathbf{s}(\mathbf{r},\omega), \quad \text{where} \quad \mathbf{s}(\mathbf{r},\omega) = \begin{Bmatrix} e^{-ik_0x/L} \\ e^{ik_0x/L} \end{Bmatrix} \\ &\text{and} \quad \mathbf{\Gamma}(\omega) = \frac{1}{1 - e^{-i2k_0}}\begin{bmatrix} 1 & -e^{-2ik_0} \\ -e^{-ik_0} & e^{-ik_0} \end{bmatrix} \end{aligned} \qquad (34)$$

Now we need to substitute $\mathbf{s}(\mathbf{r},\omega)$ in equation (25) and (26) to obtain the deterministic and random part of the element matrices. In this paper a deterministic constant modal damping factor is assumed. Therefore, we will only derive the mass and stiffness matrices of the system. For the axial vibration the stiffness operator $\mathcal{L}_2(\bullet) = \frac{\partial(\bullet)}{\partial x}$. Because constant nominal values are assumed, we have $k_{s_0}(\mathbf{r}) = A_0E_0$. Using these, from equation (25) one obtains

$$\widetilde{\mathbf{K}}_{0e}(\omega) = A_0E_0\int_0^L \left\{\frac{\partial \mathbf{s}(x,\omega)}{\partial x}\right\}\left\{\frac{\partial \mathbf{s}(x,\omega)}{\partial x}\right\}^T \mathrm{d}x \qquad (35)$$

$$= \frac{A_0E_0k_0}{L}\begin{bmatrix} -1/2\,i\left(-1 + \mathrm{e}^{-2\,ik_0}\right) & k_0 \\ k_0 & 1/2\,i\left(\mathrm{e}^{2\,ik_0} - 1\right) \end{bmatrix} \qquad (36)$$

The deterministic part of the stiffness matrix can be obtained from equation (15) using the $\mathbf{\Gamma}(\omega)$ matrix defined in equation (34). The term $\widetilde{\mathbf{M}}_{0e}(\omega)$ can be obtained in a similar way as

$$\widetilde{\mathbf{M}}_{0e}(\omega) = m_0\int_0^L \mathbf{s}(x,\omega)\mathbf{s}^T(x,\omega)\mathrm{d}x \qquad (37)$$

$$= m_0L\begin{bmatrix} \frac{1/2\,i\left(-1+\mathrm{e}^{-2\,ik_0}\right)}{k_0} & 1 \\ 1 & \frac{-1/2\,i\left(\mathrm{e}^{2\,ik_0}-1\right)}{k_0} \end{bmatrix} \qquad (38)$$

The deterministic mass matrix can be obtained from the above equation as $\mathbf{M}_{0e}(\omega) = \mathbf{\Gamma}(\omega)\widetilde{\mathbf{M}}_{0e}(\omega)\mathbf{\Gamma}^T(\omega)$.

To obtain the matrices associated with the random components, note that for each j there will be two different matrices corresponding to the two eigenfunctions defined in equations (5) and (7). Following equation (18), we can express the element stiffness matrix as

$$\mathbf{K}_e(\omega,\theta) = \mathbf{K}_{0e}(\omega) + \Delta\mathbf{K}_e(\omega) \qquad (39)$$

where $\Delta\mathbf{K}_e(\omega)$ is the random part of the matrix. Following equation (24), this matrix can be conveniently expressed as

$$\Delta\mathbf{K}_e(\omega) = \mathbf{\Gamma}(\omega)\widetilde{\Delta\mathbf{K}}_e(\omega)\mathbf{\Gamma}^T(\omega) \qquad (40)$$

The matrix $\widetilde{\Delta\mathbf{K}}_e(\omega)$ can be expanded utilizing the Karhunen-Loève expansion as

$$\widetilde{\Delta\mathbf{K}}_e(\omega) = \sum_{j=1}^{N_{\mathrm{K}}}\left[\xi_{\mathrm{K}_j}(\theta)\sqrt{\lambda_{\mathrm{K}_j}}\widetilde{\mathbf{K}}_{je}(\omega) + \xi^*_{\mathrm{K}_j}(\theta)\sqrt{\lambda^*_{\mathrm{K}_j}}\widetilde{\mathbf{K}^*}_{je}(\omega)\right] \qquad (41)$$

where $\sqrt{\lambda_{\mathrm{K}_j}}$, $\sqrt{\lambda^*_{\mathrm{K}_j}}$ are the eigenvalues corresponding to the random field $H_{AE}(x,\theta)$. The matrices $\widetilde{\mathbf{K}}_{je}(\omega)$ and $\widetilde{\mathbf{K}^*}_{je}(\omega)$ can be obtained using the integrals of the form equation (26). Using the expression of the eigenfunction for the odd values

of j as in equation (5) one has

$$\widetilde{\mathbf{K}}_{je}(\omega) = \int_0^L \frac{\epsilon_{AE} A_0 E_0 \cos(\omega_j x)}{\sqrt{a + \frac{\sin(2\omega_j a)}{2\omega_j}}} \left\{\frac{\partial \mathbf{s}(x,\omega)}{\partial x}\right\} \left\{\frac{\partial \mathbf{s}(x,\omega)}{\partial x}\right\}^T \mathrm{d}x \tag{42}$$

$$= -\frac{\epsilon_{AE} A_0 E_0 k_0^2}{L^2 \sqrt{a + \frac{\sin(2\omega_j a)}{2\omega_j}}} \int_0^L \cos(\omega_j x) \begin{bmatrix} \mathrm{e}^{\frac{-2ik_0 x}{L}} & 1 \\ 1 & \mathrm{e}^{\frac{2ik_0 x}{L}} \end{bmatrix} \mathrm{d}x \tag{43}$$

In the above expression $a = L/2$ and the eigenvalues ω_j should be obtained by solving the transcendental equation (6). In a similar manner, using the expression of the eigenfunction for the even values of j as in equation (7) one has

$$\widetilde{\mathbf{K}}^*_{je}(\omega) = -\frac{\epsilon_{AE} A_0 E_0 k_0^2}{L^2 \sqrt{a - \frac{\sin(2\omega_j a)}{2\omega_j}}} \int_0^L \sin(\omega_j x) \begin{bmatrix} \mathrm{e}^{\frac{-2ik_0 x}{L}} & 1 \\ 1 & \mathrm{e}^{\frac{2ik_0 x}{L}} \end{bmatrix} \mathrm{d}x \tag{44}$$

The mass matrix can also be represented as equations (39)–(41). The eigenvalues and eigenfunctions corresponding to the random field $H_m(x,\theta)$ needs to be used to obtain the elements of $\widetilde{\mathbf{M}}_{je}(\omega)$ and $\widetilde{\mathbf{M}}^*_{je}(\omega)$.

Using the expression of the eigenfunction for the odd values of j as in equation (5) one has

$$\widetilde{\mathbf{M}}_{je}(\omega) = \int_0^L \frac{\epsilon_m m_0 \cos(\omega_j x)}{\sqrt{a + \frac{\sin(2\omega_j a)}{2\omega_j}}} \mathbf{s}(x,\omega)\mathbf{s}^T(x,\omega)\mathrm{d}x \tag{45}$$

$$= \frac{\epsilon_m m_0}{\sqrt{a + \frac{\sin(2\omega_j a)}{2\omega_j}}} \int_0^L \cos(\omega_j x) \begin{bmatrix} \mathrm{e}^{\frac{-2ik_0 x}{L}} & 1 \\ 1 & \mathrm{e}^{\frac{2ik_0 x}{L}} \end{bmatrix} \mathrm{d}x \tag{46}$$

In the above expression the eigenvalues ω_j should be obtained by solving the transcendental equation (6). In a similar manner, using the expression of the eigenfunction for the even values of j as in equation (7) one has

$$\widetilde{\mathbf{M}}^*_{je}(\omega) = \frac{\epsilon_m m_0}{\sqrt{a - \frac{\sin(2\omega_j a)}{2\omega_j}}} \int_0^L \sin(\omega_j x) \begin{bmatrix} \mathrm{e}^{\frac{-2ik_0 x}{L}} & 1 \\ 1 & \mathrm{e}^{\frac{2ik_0 x}{L}} \end{bmatrix} \mathrm{d}x \tag{47}$$

Equations (42)–(47) completely define the random parts of the element stiffness and mass matrices. The definite integrals appearing in these expressions can be evaluated in closed-form. This further reduces the computational cost in deriving the element matrices. The exact closed-form expression of the elements of the above four matrices are given in the appendix.

6. CONCLUSIONS

The basic principles for Doubly Spectral Stochastic Finite Element Method (DSSFEM) for damped linear dynamical systems with distributed parametric uncertainty has been derived. This new approach simultaneously utilizes the spectral representations in the frequency and random domains. The spatial displacement fields are discretized using frequency-adaptive complex shape functions while the spatial random fields are discretized using the Karhunen-Loève expansion. In spite of the fact that these two spectral approaches existed for well over three decades, there has been very little overlap between them in literature. In this paper these two spectral techniques have been unified with the aim that the unified approach would outperform any of the spectral methods considered on its own. The resulting frequency depended random element matrices in general turn out to be complex symmetric matrices. The main computational advantage of the proposed approach is that the fine spatial discretisation will not be necessary for high and mid-frequency vibration analysis. The detailed derivations for rods in axial vibration is given. Closed-form expressions of the element mass and stiffness matrices have been derived for the stochastic parametric fields with exponential autocorrelation function. Numerical examples have been given to illustrate the applicability of the proposed method.

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APPENDIX A: DERIVATION OF THE RANDOM ELEMENT STIFFNESS AND MASS MATRICES

In this section we derive the exact closed-form expressions of the different terms of the element stiffness and mass matrices corresponding to the random part of the respective matrices. The definite integral which define these matrices are given in equations (42)–(47). Since all of these matrices are symmetric, only the expressions of the elements in the upper triangular part are given here.

We express the matrix $\widetilde{\mathbf{K}}_{je}(\omega)$ in equation (42) as

$$\widetilde{\mathbf{K}}_{je}(\omega) = \frac{\epsilon_{AE} A_0 E_0 k_0^2}{L\sqrt{a + \dfrac{\sin(2\omega_j a)}{2\omega_j}}} \widehat{\mathbf{K}}_{je}(\omega) \tag{48}$$

The elements of the upper triangular part of the matrix

$$\widehat{\mathbf{K}}_{je}(\omega) = -\frac{1}{L}\int_0^L \cos(\omega_j x) \begin{bmatrix} \mathrm{e}^{\frac{-2ik_0 x}{L}} & 1 \\ 1 & \mathrm{e}^{\frac{2ik_0 x}{L}} \end{bmatrix} \mathrm{d}x \tag{49}$$

are given by

$$\widehat{K}_{je_{11}}(\omega) = -\frac{2\,ik_0 - 2\,ie^{-2\,ik_0}k_0\cos\left(\omega_j L\right) + e^{-2\,ik_0}\omega_j L\sin\left(\omega_j L\right)}{-4\,{k_0}^2 + {\omega_j}^2 L^2}$$

$$\widehat{K}_{je_{12}}(\omega) = \frac{\sin\left(\omega_j L\right)}{\omega_j L}$$

$$\widehat{K}_{je_{22}}(\omega) = -\frac{-2\,ik_0 + 2\,ie^{2\,ik_0}k_0\cos\left(\omega_j L\right) + e^{2\,ik_0}L\omega_j\sin\left(\omega_j L\right)}{-4\,{k_0}^2 + {\omega_j}^2 L^2} \tag{50}$$

Similarly the matrix $\widetilde{\mathbf{K}^*}_{je}(\omega)$ in equation (42) can be expressed as

$$\widetilde{\mathbf{K}^*}_{je}(\omega) = \frac{\epsilon_{AE}A_0E_0k_0^2}{L\sqrt{a - \dfrac{\sin(2\omega_j a)}{2\omega_j}}}\widehat{\mathbf{K}^*}_{je}(\omega) \tag{51}$$

The elements of the upper triangular part of the matrix

$$\widehat{\mathbf{K}^*}_{je}(\omega) = -\frac{1}{L}\int_0^L \sin(\omega_j x)\begin{bmatrix} e^{\frac{-2\,ik_0x}{L}} & 1 \\ 1 & e^{\frac{2\,ik_0x}{L}} \end{bmatrix}\mathrm{d}x \tag{52}$$

are given by

$$\widehat{K^*}_{je_{11}}(\omega) = \frac{-\omega_j L + e^{-2\,ik_0}\omega_j L\cos\left(\omega_j L\right) + 2\,ie^{-2\,ik_0}k_0\sin\left(\omega_j L\right)}{-4\,{k_0}^2 + {\omega_j}^2 L^2}$$

$$\widehat{K^*}_{je_{12}}(\omega) = -\frac{-1 + \cos\left(\omega_j L\right)}{\omega_j L}$$

$$\widehat{K^*}_{je_{22}}(\omega) = \frac{-\omega_j L + e^{2\,ik_0}\omega_j L\cos\left(\omega_j L\right) - 2\,ie^{2\,ik_0}k_0\sin\left(\omega_j L\right)}{-4\,{k_0}^2 + {\omega_j}^2 L^2} \tag{53}$$

The matrices associated with the random parts of the mass matrix given by equations (46) and (47) can be obtained in closed-form by using the integrals given by equations (49) and (52) respectively.

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