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Calculation of Eigensolution Derivatives for Nonviscously Damped Systems Using Nelson's Method

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A method to calculate the derivatives of the eigenvalues and eigenvectors of multiple-degree-of-freedom damped linear dynamic systems with respect to arbitrary design parameters is presented. In contrast to the traditional viscous damping model, a more general nonviscous damping model is considered. The nonviscous damping model is such that the damping forces depend on the past history of velocities via convolution integrals over given kernel functions. Because of the general nature of the damping, eigensolutions are generally complex valued, and eigenvectors do not satisfy the classical orthogonality relationship. The proposed method to calculate the eigenvector derivative depends only on the eigenvector concerned. Numerical examples are provided to illustrate the derived results.

Nomenclature

c_j	=	constant associated with the derivative of $\boldsymbol{u}_j$
$\boldsymbol{D}(s)$	=	dynamic stiffness matrix
d_j	=	constant associated with the derivative of $\boldsymbol{v}_j$
$\boldsymbol{G}(s)$	=	Laplace transform of $\boldsymbol{\mathcal{G}}(t)$
$\boldsymbol{\mathcal{G}}(t)$	=	nonviscous damping matrix
$\boldsymbol{K}$	=	stiffness matrix
$\mathcal{L}[\,]$	=	Laplace transform of [ ]
$\boldsymbol{M}$	=	mass matrix
m	=	total number of eigenvectors
N	=	number of degrees of freedom of the system
n	=	number of relaxation parameters
p	=	design parameter
rank( )	=	rank of a matrix
s	=	Laplace domain parameter
$\boldsymbol{u}_j$	=	jth eigenvector
$u(t)$	=	displacement vector
$\boldsymbol{v}_j$	=	jth adjoint (left) eigenvector
$\boldsymbol{x}_j$	=	vector associated with the derivative of $\boldsymbol{u}_j$
$\boldsymbol{y}_j$	=	vector associated with the derivative of $\boldsymbol{v}_j$
θ_j	=	normalization constant for the jth eigenvector
λ_j	=	jth eigenvalue
μ_k	=	relaxation parameters

Subscripts

$(\,)^T$	=	matrix transpose
$(\,)'$	=	derivative with respect to s

I. Introduction

THE calculation of derivatives of natural frequencies and mode shapes with respect to model parameters is vital for design optimization,[1] model updating,[2] probabilistic structural dynamics,[3] and many other applications. The methods to calculate these derivatives are well established for undamped structures. Fox and Kapoor[4] calculated the derivative of the eigenvectors by expressing these derivatives as a linear combination of the undamped eigenvectors. The expressions derived in Ref. 4 are valid for symmetric undamped systems. Many authors[5–8] have extended Fox and Kapoor's[4] approach to determine eigensolution derivatives to more general asymmetric systems. Nelson[9] introduced the approach, extended in this paper, where only the eigenvector of interest was required.

The works just discussed do not explicitly consider the damping present in the system. To apply these results to systems with general nonproportional (viscous) damping, the equations of motion must be converted into the state-space form (see Ref. 10, for example). Although exact in nature, the state-space methods require significant numerical effort as the size of the matrices doubles. Moreover, these methods also lack some of the intuitive simplicity of the analysis based on N space. For these reasons some authors have considered the problem of calculating the derivatives of the eigensolutions of viscously damped systems in N space. One of the earliest papers to consider damping was by Cardani and Mantegazza[11] in the context of flutter problems. Note that unlike undamped systems, in damped systems the eigenvalues and eigenvectors, and consequently their derivatives, become complex in general. Adhikari[12] derived an exact expression for the derivative of complex eigenvalues and eigenvectors. The results were expressed in terms of the complex eigenvalues and eigenvectors of the second-order system, and the state-space representation of the equation of motion was avoided. Lee et al.[13,14] proposed an approach to determine natural frequency and mode-shape sensitivities of damped systems. Adhikari[15] suggested an approximate method to calculate the derivative of complex modes using a modal series involving only classical normal modes. Friswell and Adhikari[16] extended Nelson's method to nonproportionally damped systems with complex modes. Later Adhikari and Friswell[17] derived the first- and second-order derivatives of complex eigensolutions for more general asymmetric nonconservative systems. Recently Choi et al.[18] proposed a matrix inversion approach to determine mode-shape sensitivities of damped systems.

The preceding studies only considered a viscous damping model. However, it is well known that other damping models exist within the scope of linear analysis, such as damping in composite materials,[19] energy dissipation in structural joints,[20,21] and damping mechanisms in composite beams.[22] We consider a class of nonviscous damping models in which the damping forces depend on the past history of motion via convolution integrals over some kernel functions. The equations of motion describing free vibration of a N-degree-of-freedom linear system with such damping can be expressed by

$$\boldsymbol{M}\ddot{u}(t) + \int_{-\infty}^{t} \boldsymbol{\mathcal{G}}(t-\tau)\,\dot{u}(\tau)\,\mathrm{d}\tau + \boldsymbol{K}u(t) = 0 \qquad (1)$$

Here $\boldsymbol{M}$ and $\boldsymbol{K} \in \mathbb{R}^{N \times N}$ are the mass and stiffness matrices, $\boldsymbol{\mathcal{G}}(t) \in \mathbb{R}^{N \times N}$ is the matrix of kernel functions, and 0 is an $N \times 1$

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vector of zeros. In the special case when $\mathcal{G}(t-\tau)=\boldsymbol{C}\,\delta(t-\tau)$, Eq. (1) reduces to the case of a viscously damped system. The damping model of this kind is a further generalization of the familiar viscous damping. Dynamic analysis of nonviscously damped systems has been discussed in detail in Refs. 23–26. Dynamic systems mathematically equivalent to Eq. (1) have been used by several authors in the context of viscoelasticity damped systems. Examples include, but are not limited to, the fractional derivative approach,[27] the Golla–Hughes–McTavish approach,[28,29] and the approach proposed by Lesieutre and Mingori[30] and Lesieutre and Bianchini.[31] In many of these methods, after some manipulation, the eigenvalue problem associated with Eq. (1) can be cast in the form of a λ-matrix problem.[32] Derivatives of eigenvalues and eigenvectors for this general case have been discussed rigorously in Refs. 33–35. Adhikari[36] extended the method of Fox and Kapoor to systems with nonviscous damping in the form of Eq. (1). In this method the eigenvector derivative was obtained as a linear combination of all of the eigenvectors. For large-scale structures with nonviscous damping, obtaining all of the eigenvectors is a computationally expensive task because the number of eigenvectors of a nonviscously damped system is much larger, in general, than the number for a viscously damped system. This motivates the extension of Nelson's method to calculate the derivatives of eigenvectors of nonviscously damped systems. We begin by considering the self-adjoint case, where the matrices are symmetric, and in Sec. VI consider the general case.

II. Eigenvalues and Eigenvectors

The determination of eigenvalues and eigenvectors of nonviscously damped systems has been discussed by Adhikari.[23] Here we briefly outline the topics required for further developments. Taking the Laplace transform of Eq. (1), we have

$$s^2\boldsymbol{M}\bar{\boldsymbol{u}}(s)+s\,\boldsymbol{G}(s)\bar{\boldsymbol{u}}(s)+\boldsymbol{K}\bar{\boldsymbol{u}}(s)=0 \quad \text{or} \quad \boldsymbol{D}(s)\bar{\boldsymbol{u}}(s)=0 \tag{2}$$

where the dynamic stiffness matrix is

$$\boldsymbol{D}(s)=[s^2\boldsymbol{M}+s\,\boldsymbol{G}(s)+\boldsymbol{K}]\ \in\mathbb{C}^{N\times N} \tag{3}$$

Here $\bar{\boldsymbol{u}}(s)=\mathcal{L}[u(t)]\in\mathbb{C}^N$, $\boldsymbol{G}(s)=\mathcal{L}[\mathcal{G}(t)]\in\mathbb{C}^{N\times N}$ and $\mathcal{L}[\,]$ denotes the Laplace transform. In the context of structural dynamics, $s=i\omega$, where $\omega\in\mathbb{R}^+$ denotes the frequency. We consider the damping to be nonproportional (Adhikari[37] derived conditions for proportionality for structures with nonviscous damping), so that the mass and stiffness matrices, and also the matrix of kernel functions, cannot be simultaneously diagonalized by any linear transformation. However, it is assumed that $\boldsymbol{M}^{-1}$ exists and $\boldsymbol{G}(s)$ is such that the motion is dissipative. Conditions that $\boldsymbol{G}(s)$ must satisfy in order to produce dissipative motion were given by Golla and Hughes.[28]

The eigenvalue problem associated with Eq. (1) can be defined from Eq. (2) as

$$\left[\lambda_j^2\boldsymbol{M}+\lambda_j\,\boldsymbol{G}(\lambda_j)+\boldsymbol{K}\right]\boldsymbol{u}_j=0 \tag{4}$$

or

$$\boldsymbol{D}(\lambda_j)\boldsymbol{u}_j=0 \tag{5}$$

where $\boldsymbol{u}_j\in\mathbb{C}^N$ is the jth eigenvector. The eigenvalues λ_j are roots of the characteristic equation

$$\det[s^2\boldsymbol{M}+s\,\boldsymbol{G}(s)+\boldsymbol{K}]=0 \tag{6}$$

For the linear viscoelastic case it can be shown that,[38,39] in general, the elements of $\boldsymbol{G}(s)$ can be represented by

$$G_{jk}(s)=\frac{p_{jk}(s)}{q_{jk}(s)} \tag{7}$$

where $p_{jk}(s)$ and $q_{jk}(s)$ are finite-order polynomials in s and the degree of $p_{jk}(s)$ is not more than that of $q_{jk}(s)$. Under such assumptions the order of the characteristic equation m is usually more than $2N$. Thus, although the system has N degrees of freedom, the number of eigenvalues is more than $2N$. This is a major difference between nonviscously damped systems and viscously damped systems where the number of eigenvalues is exactly $2N$, including any multiplicities. A recent study on a single-degree-of-freedom system[40] has explored the nature of the eigenvalues in detail. For multiple-degree-of-freedom systems, one can group the eigenvectors[23] as 1) elastic modes (corresponding to N complex conjugate pairs of eigenvalues) and 2) nonviscous modes (corresponding to the additional $m-2N$ eigenvalues). The elastic modes are related to the N modes of vibration of the structural system. In this paper we assume that all m eigenvalues are distinct.

Adhikari[24] discussed the orthogonality and the normalization relationships of the eigenvectors. Noting the symmetry of $\boldsymbol{D}(s)$ and using Eq. (5), we can obtain

$$\boldsymbol{u}_j^T[\boldsymbol{D}(\lambda_k)-\boldsymbol{D}(\lambda_j)]\boldsymbol{u}_k=0 \tag{8}$$

This equation can be regarded as the orthogonality relationship of the eigenvectors. It is easy to verify that, in the undamped limit, Eq. (8) degenerates to the familiar mass orthogonality relationship of the undamped eigenvectors.

There are many approaches to the normalization of the eigenvectors. A convenient approach is to normalize $\boldsymbol{u}_j$ such that

$$\boldsymbol{u}_j^T\left.\frac{\partial\boldsymbol{D}(s)}{\partial s}\right|_{s=\lambda_j}\boldsymbol{u}_j=\theta_j \tag{9}$$

or

$$\boldsymbol{u}_j^T\boldsymbol{D}'(\lambda_j)\boldsymbol{u}_j=\theta_j,\qquad \forall j=1,\ldots,m \tag{10}$$

where

$$\boldsymbol{D}'(s)=\frac{\partial\boldsymbol{D}(s)}{\partial s}=[2s\boldsymbol{M}+\boldsymbol{G}(s)+s\boldsymbol{G}'(s)]\in\mathbb{C}^{N\times N} \tag{11}$$

and $\theta_j\in\mathbb{C}$ is some nonzero constant. Equation (10) reduces to the corresponding normalization relationship for viscously damped systems (for example, see Refs. 41 and 42) when $\boldsymbol{G}(s)$ is constant with respect to s. Numerical values of θ_j can be selected in various ways. For example, one can choose $\theta_j=2\lambda_j,\forall j$, which reduces to $\boldsymbol{u}_j^T\boldsymbol{M}\boldsymbol{u}_j=1$ when the damping is zero. This is consistent with the familiar unity modal mass convention. Alternatively, one can choose $\theta_j=1,\forall j$, and any theoretical analysis is easiest with this normalization. Alternatively, one can choose other kinds of normalization, for example, $max(\boldsymbol{u}_j)=1$ or $|\boldsymbol{u}_j|=1$. In such cases one simply calculates the numerical values of θ_j from Eq. (10) and uses them subsequently. Further discussions on the normalization of complex eigenvectors can be found in Refs. 24 and 43.

III. Eigenvalue Derivatives

The derivatives of the eigenvalues were obtained by Adhikari,[36] and we provide a brief review here for completeness. Suppose the system matrices in Eq. (1) are functions of some design parameter p. In this section we intend to obtain an expression for the derivative of the jth eigenvalue with respect to the design parameter p. Differentiating Eq. (4) with respect to p, one obtains

$$\left[2\lambda_j\frac{\partial\lambda_j}{\partial p}\boldsymbol{M}+\lambda_j^2\frac{\partial\boldsymbol{M}}{\partial p}+\frac{\partial\lambda_j}{\partial p}\boldsymbol{G}(\lambda_j)+\lambda_j\frac{\partial[\boldsymbol{G}(\lambda_j)]}{\partial p}+\frac{\partial\boldsymbol{K}}{\partial p}\right]\boldsymbol{u}_j$$
$$+\left[\lambda_j^2\boldsymbol{M}+\lambda_j\,\boldsymbol{G}(\lambda_j)+\boldsymbol{K}\right]\frac{\partial\boldsymbol{u}_j}{\partial p}=0 \tag{12}$$

The term $\partial[\boldsymbol{G}(\lambda_j)]/\partial p$ in the preceding equation can be expressed as

$$\frac{\partial[\boldsymbol{G}(\lambda_j)]}{\partial p}=\frac{\partial\lambda_j}{\partial p}\left.\frac{\partial\boldsymbol{G}(s)}{\partial s}\right|_{s=\lambda_j}+\left.\frac{\partial\boldsymbol{G}(s)}{\partial p}\right|_{s=\lambda_j} \tag{13}$$

Premultiplying Eq. (12) by $\boldsymbol{u}_j^T$ and using the symmetry property of the system matrices, it can be observed that the second term vanishes as a result of Eq. (4). Substituting Eq. (13) into Eq. (12), we obtain

$$\boldsymbol{u}_j^T\left[\lambda_j^2\frac{\partial \boldsymbol{M}}{\partial p}+\lambda_j\frac{\partial \boldsymbol{G}(s)}{\partial p}\bigg|_{s=\lambda_j}+\frac{\partial \boldsymbol{K}}{\partial p}\right]\boldsymbol{u}_j+\boldsymbol{u}_j^T\left[2\lambda_j\frac{\partial \lambda_j}{\partial p}\boldsymbol{M}+\right.$$
$$\left.+\frac{\partial \lambda_j}{\partial p}\boldsymbol{G}(\lambda_j)+\lambda_j\frac{\partial \lambda_j}{\partial p}\frac{\partial \boldsymbol{G}(s)}{\partial s}\bigg|_{s=\lambda_j}\right]\boldsymbol{u}_j=0 \tag{14}$$

Rearranging this equation gives the derivative of the eigenvalues as

$$\frac{\partial \lambda_j}{\partial p}=-\boldsymbol{u}_j^T\left[\lambda_j^2\frac{\partial \boldsymbol{M}}{\partial p}+\lambda_j\frac{\partial \boldsymbol{G}(s)}{\partial p}\bigg|_{s=\lambda_j}+\frac{\partial \boldsymbol{K}}{\partial p}\right]\boldsymbol{u}_j\Bigg/$$
$$\boldsymbol{u}_j^T\left[2\lambda_j\boldsymbol{M}+\boldsymbol{G}(\lambda_j)+\lambda_j\frac{\partial \boldsymbol{G}(s)}{\partial s}\bigg|_{s=\lambda_j}\right]\boldsymbol{u}_j \tag{15}$$

The denominator of Eq. (15) is exactly the normalization relationship given by Eq. (10). Using the expression of the dynamic stiffness matrix in Eq. (3), one can easily deduce that

$$\frac{\partial \boldsymbol{D}(s)}{\partial p}\bigg|_{s=\lambda_j}=\left[\lambda_j^2\frac{\partial \boldsymbol{M}}{\partial p}+\lambda_j\frac{\partial \boldsymbol{G}(s)}{\partial p}\bigg|_{s=\lambda_j}+\frac{\partial \boldsymbol{K}}{\partial p}\right] \tag{16}$$

Using these relationships, Eq. (15) can be expressed in a concise form as

$$\frac{\partial \lambda_j}{\partial p}=-\boldsymbol{u}_j^T\frac{\partial \boldsymbol{D}(s)}{\partial p}\bigg|_{s=\lambda_j}\boldsymbol{u}_j\Bigg/\boldsymbol{u}_j^T\frac{\partial \boldsymbol{D}(s)}{\partial s}\bigg|_{s=\lambda_j}\boldsymbol{u}_j$$

or

$$\frac{\partial \lambda_j}{\partial p}=-\frac{1}{\theta_j}\left[\boldsymbol{u}_j^T\frac{\partial \boldsymbol{D}(s)}{\partial p}\bigg|_{s=\lambda_j}\boldsymbol{u}_j\right] \tag{17}$$

This is the most general expression for the derivative of the eigenvalues of linear dynamic systems. Some special cases of this expression are given in Sec. V. The derivatives of the associated eigenvectors are considered in the next section.

IV. Eigenvector Derivatives

Adhikari[36] has shown that the derivative of the eigenvectors can be expressed by a linear combination of all of the eigenvectors as

$$\frac{\partial \boldsymbol{u}_j}{\partial p}=-\frac{1}{2\theta_j}\left(\boldsymbol{u}_j^T\frac{\partial \boldsymbol{D}'(s)}{\partial p}\bigg|_{s=\lambda_j}\boldsymbol{u}_j\right)\boldsymbol{u}_j$$
$$-\sum_{\substack{k=1\\k\neq j}}^{m}\left[\boldsymbol{u}_k^T\frac{\partial \boldsymbol{D}(s)}{\partial p}\bigg|_{s=\lambda_j}\boldsymbol{u}_j\Bigg/\theta_k(\lambda_j-\lambda_k)\right]\boldsymbol{u}_k \tag{18}$$

where

$$\frac{\partial \boldsymbol{D}'(s)}{\partial p}\bigg|_{s=\lambda_j}=2\lambda_j\frac{\partial \boldsymbol{M}}{\partial p}+\frac{\partial \boldsymbol{G}(s)}{\partial p}\bigg|_{s=\lambda_j}+\lambda_j\frac{\partial \boldsymbol{G}'(s)}{\partial p}\bigg|_{s=\lambda_j} \tag{19}$$

For complex nonviscously damped systems the total number of eigenvectors m can be very large ($m\gg 2N$). The calculation of all eigenvectors to obtain the derivatives of only few eigenvectors can be computationally expensive. In this section Nelson's method is extended to nonviscously damped systems, which does not suffer from this drawback.

Differentiating Eq. (5) with respect to the design parameter p, we have

$$\boldsymbol{D}(\lambda_j)\frac{\partial \boldsymbol{u}_j}{\partial p}=\boldsymbol{h}_j \tag{20}$$

where

$$\boldsymbol{h}_j=-\frac{\partial \boldsymbol{D}(\lambda_j)}{\partial p}\boldsymbol{u}_j=-\left\{2\lambda_j\frac{\partial \lambda_j}{\partial p}\boldsymbol{M}+\lambda_j^2\frac{\partial \boldsymbol{M}}{\partial p}+\frac{\partial \lambda_j}{\partial p}\boldsymbol{G}(\lambda_j)\right.$$
$$\left.+\lambda_j\frac{\partial[\boldsymbol{G}(\lambda_j)]}{\partial p}+\frac{\partial \boldsymbol{K}}{\partial p}\right\}\boldsymbol{u}_j \tag{21}$$

is known. Equation (20) cannot be solved to obtain the eigenvector derivative because the matrix is singular. For distinct eigenvalues this matrix has a null space of dimension one. Following Nelson's approach, the eigenvector derivative is written as

$$\frac{\partial \boldsymbol{u}_j}{\partial p}=\boldsymbol{x}_j+c_j\boldsymbol{u}_j \tag{22}$$

where $\boldsymbol{x}_j$ and c_j have to be determined. These quantities are not unique because any multiple of the eigenvector can be added to $\boldsymbol{x}_j$. A convenient choice is to identify the element of maximum magnitude in $\boldsymbol{u}_j$ and make the corresponding element in $\boldsymbol{x}_j$ equal to zero. Although other elements of $\boldsymbol{x}_j$ could be set to zero, this choice is most likely to produce a numerically well-conditioned problem. Because $\boldsymbol{D}(\lambda_j)\boldsymbol{u}_j=\boldsymbol{0}$ as a result of Eq. (5), substituting Eq. (22) into Eq. (20) gives

$$\boldsymbol{D}_j\boldsymbol{x}_j=\boldsymbol{h}_j \tag{23}$$

where

$$\boldsymbol{D}_j=\boldsymbol{D}(\lambda_j)=\left[\lambda_j^2\boldsymbol{M}+\lambda_j\boldsymbol{G}(\lambda_j)+\boldsymbol{K}\right]\in\mathbb{C}^{N\times N} \tag{24}$$

This can be solved, including the constraint on the zero element of $\boldsymbol{x}_j$ by solving the equivalent problem,

$$\begin{bmatrix}\boldsymbol{D}_{j11}&0&\boldsymbol{D}_{j31}\\0&1&0\\\boldsymbol{D}_{j31}&0&\boldsymbol{D}_{j33}\end{bmatrix}\begin{Bmatrix}\boldsymbol{x}_{j1}\\x_{j2}(=0)\\\boldsymbol{x}_{j3}\end{Bmatrix}=\begin{Bmatrix}\boldsymbol{h}_{j1}\\0\\\boldsymbol{h}_{j3}\end{Bmatrix} \tag{25}$$

where the $\boldsymbol{D}_j$ is defined in Eq. (24), and has the row and column corresponding to the zeroed element of $\boldsymbol{x}_j$ replaced with the corresponding row and column of the identity matrix. This approach maintains the banded nature of the structural matrices and hence is computationally efficient.

It only remains to compute the scalar constant c_j to obtain the eigenvector derivative. For this the normalization equation (10) must be used. Differentiating Eq. (10) and using the symmetry property of $\boldsymbol{D}'(s)$, we have

$$\boldsymbol{u}_j^T\frac{\partial \boldsymbol{D}'(\lambda_j)}{\partial p}\boldsymbol{u}_j+2\boldsymbol{u}_j^T\boldsymbol{D}'(\lambda_j)\frac{\partial \boldsymbol{u}_j}{\partial p}=0 \tag{26}$$

Substituting $\partial\boldsymbol{u}_j/\partial p$ from Eq. (22), one has

$$\frac{1}{2}\boldsymbol{u}_j^T\frac{\partial \boldsymbol{D}'(\lambda_j)}{\partial p}\boldsymbol{u}_j+\boldsymbol{x}_j^T\boldsymbol{D}'(\lambda_j)\boldsymbol{u}_j+c_j\boldsymbol{u}_j^T\boldsymbol{D}'(\lambda_j)\boldsymbol{u}_j=0 \tag{27}$$

Noticing that the coefficient associated with c_j is the normalization constant given by Eq. (10), we have

$$c_j=-\frac{1}{\theta_j}\left\{\frac{1}{2}\boldsymbol{u}_j^T\frac{\partial \boldsymbol{D}'(\lambda_j)}{\partial p}\boldsymbol{u}_j+\boldsymbol{u}_j^T\boldsymbol{D}'(\lambda_j)\boldsymbol{x}_j\right\} \tag{28}$$

The first term on the right-hand side can be obtained by substituting $s=\lambda_j$ into Eq. (11) and differentiating

$$\frac{\partial \boldsymbol{D}'(\lambda_j)}{\partial p}=2\frac{\partial \lambda_j}{\partial p}\boldsymbol{M}+2\lambda_j\frac{\partial \boldsymbol{M}}{\partial p}+\frac{\partial[\boldsymbol{G}(\lambda_j)]}{\partial p}$$
$$+\frac{\partial \lambda_j}{\partial p}\boldsymbol{G}'(\lambda_j)+\lambda_j\frac{\partial[\boldsymbol{G}'(\lambda_j)]}{\partial p} \tag{29}$$

where $\partial[\boldsymbol{G}(\lambda_j)]/\partial p$ is given in Eq. (13) and

$$\frac{\partial[\boldsymbol{G}'(\lambda_j)]}{\partial p} = \frac{\partial\lambda_j}{\partial p}\frac{\partial \boldsymbol{G}'(s)}{\partial s}\bigg|_{s=\lambda_j} + \frac{\partial \boldsymbol{G}'(s)}{\partial p}\bigg|_{s=\lambda_j}$$

$$= \frac{\partial\lambda_j}{\partial p}\frac{\partial^2 \boldsymbol{G}(\boldsymbol{s})}{\partial s^2}\bigg|_{s=\lambda_j} + \frac{\partial^2 \boldsymbol{G}(s)}{\partial p\,\partial s}\bigg|_{s=\lambda_j} \tag{30}$$

Equation (22) combined with $\boldsymbol{x}_j$ obtained by solving Eq. (25) and c_j obtained from Eq. (28) completely defines the derivative of the eigenvectors.

V. Special Cases

The expressions for the derivative of the eigenvalues and eigenvectors proposed in this paper are quite general in their scope. In this section some special cases of these expressions are derived to relate them to the published literature.

A. Undamped Systems

The derivatives of undamped eigensolutions have been considered, for example, by Fox and Kapoor[4] and Nelson.[9] In this case $\boldsymbol{G}(s) = 0,\ \forall\, s$, from which it follows that

$$\boldsymbol{D}(s) = s^2\boldsymbol{M} + \boldsymbol{K} \tag{31}$$

$$\boldsymbol{D}'(s) = 2s\boldsymbol{M} \tag{32}$$

For undamped systems all of the eigenvectors are real, that is, $\boldsymbol{u}_j \in \mathbb{R}^N,\ \forall\, j$. The number of modes, that is, the order of the characteristic polynomial, $m = 2N$. Because λ_j is purely imaginary, considering $\lambda_j = i\omega_j$, where $\omega_j \in \mathbb{R}$ is the jth undamped natural frequency, one obtains

$$\theta_j = \boldsymbol{u}_j^T\boldsymbol{D}'(\lambda_j)\boldsymbol{u}_j = 2i\omega_j\boldsymbol{u}_j^T\boldsymbol{M}\boldsymbol{u}_j \tag{33}$$

$$\frac{\partial \boldsymbol{D}(s)}{\partial p}\bigg|_{s=i\omega_j} = \left[\frac{\partial \boldsymbol{K}}{\partial p} - \omega_j^2\frac{\partial \boldsymbol{M}}{\partial p}\right] \tag{34}$$

$$\frac{\partial[\boldsymbol{D}(\lambda_j)]}{\partial p} = \left[2\lambda_j\frac{\partial\lambda_j}{\partial p}\boldsymbol{M} + \lambda_j^2\frac{\partial \boldsymbol{M}}{\partial p} + \frac{\partial \boldsymbol{K}}{\partial p}\right]$$

$$= \left[-2\omega_j\frac{\partial\omega_j}{\partial p}\boldsymbol{M} - \omega_j^2\frac{\partial \boldsymbol{M}}{\partial p} + \frac{\partial \boldsymbol{K}}{\partial p}\right] \tag{35}$$

$$\frac{\partial[\boldsymbol{D}'(\lambda_j)]}{\partial p} = 2\frac{\partial\lambda_j}{\partial p}\boldsymbol{M} + 2\lambda_j\frac{\partial \boldsymbol{M}}{\partial p} = 2i\left[\frac{\partial\omega_j}{\partial p}\boldsymbol{M} + 2\omega_j\frac{\partial \boldsymbol{M}}{\partial p}\right] \tag{36}$$

From Eq. (17) the eigenvalue derivative becomes

$$\frac{\partial\omega_j}{\partial p} = \frac{\boldsymbol{u}_j^T\left[\partial\boldsymbol{K}/\partial p - \omega_j^2(\partial\boldsymbol{M}/\partial p)\right]\boldsymbol{u}_j}{2\omega_j\boldsymbol{u}_j^T\boldsymbol{M}\boldsymbol{u}_j} \tag{37}$$

which is a well-known result. The eigenvector derivative is given by Eq. (22). The vector $\boldsymbol{h}_j$ and the constant c_j can be obtained from Eqs. (21) and (28) as

$$\boldsymbol{h}_j = \left[2\omega_j\frac{\partial\omega_j}{\partial p}\boldsymbol{M} + \omega_j^2\frac{\partial \boldsymbol{M}}{\partial p} - \frac{\partial \boldsymbol{K}}{\partial p}\right]\boldsymbol{u}_j \tag{38}$$

$$c_j = -\frac{1}{\boldsymbol{u}_j^T\boldsymbol{M}\boldsymbol{u}_j}\left\{\frac{1}{2}\boldsymbol{u}_j^T\left[\frac{1}{\omega_j}\frac{\partial\omega_j}{\partial p}\boldsymbol{M} + \frac{\partial \boldsymbol{M}}{\partial p}\right]\boldsymbol{u}_j + \boldsymbol{u}_j^T\boldsymbol{M}\boldsymbol{x}_j\right\} \tag{39}$$

B. Viscously Damped Systems

Derivatives of eigensolutions of nonproportional viscous damped systems have been considered in Refs. 12–18. In this case $\boldsymbol{G}(s) = \boldsymbol{C},\ \forall\, s$, from which it follows that

$$\boldsymbol{D}(s) = [s^2\boldsymbol{M} + s\boldsymbol{C} + \boldsymbol{K}] \tag{40}$$

$$\boldsymbol{D}'(s) = [2s\boldsymbol{M} + \boldsymbol{C}] \tag{41}$$

For viscously damped systems, the number of modes, that is, the order of the characteristic polynomial, is $m = 2N$. All of the eigenvalues and the eigenvectors appear in complex conjugate pairs. Using Eqs. (40) and (41), we obtain

$$\theta_j = \boldsymbol{u}_j^T\boldsymbol{D}'(\lambda_j)\boldsymbol{u}_j = \boldsymbol{u}_j^T[2\lambda_j\boldsymbol{M} + \boldsymbol{C}]\boldsymbol{u}_j \tag{42}$$

$$\frac{\partial \boldsymbol{D}(s)}{\partial p}\bigg|_{s=\lambda_j} = \left[\lambda_j^2\frac{\partial \boldsymbol{M}}{\partial p} + \lambda_j\frac{\partial \boldsymbol{C}}{\partial p} + \frac{\partial \boldsymbol{K}}{\partial p}\right] \tag{43}$$

$$\frac{\partial[\boldsymbol{D}(\lambda_j)]}{\partial p} = \left[2\lambda_j\frac{\partial\lambda_j}{\partial p}\boldsymbol{M} + \lambda_j^2\frac{\partial \boldsymbol{M}}{\partial p} + \frac{\partial\lambda_j}{\partial p}\boldsymbol{C} + \lambda_j\frac{\partial \boldsymbol{C}}{\partial p} + \frac{\partial \boldsymbol{K}}{\partial p}\right] \tag{44}$$

$$\frac{\partial[\boldsymbol{D}'(\lambda_j)]}{\partial p} = \left[2\frac{\partial\lambda_j}{\partial p}\boldsymbol{M} + 2\lambda_j\frac{\partial \boldsymbol{M}}{\partial p} + \frac{\partial \boldsymbol{C}}{\partial p}\right] \tag{45}$$

From Eq. (17) the eigenvalue derivative becomes

$$\frac{\partial\lambda_j}{\partial p} = -\boldsymbol{u}_j^T\left[\lambda_j^2\frac{\partial \boldsymbol{M}}{\partial p} + \lambda_j\frac{\partial \boldsymbol{C}}{\partial p} + \frac{\partial \boldsymbol{K}}{\partial p}\right]\boldsymbol{u}_j \bigg/ \boldsymbol{u}_j^T[2\lambda_j\boldsymbol{M} + \boldsymbol{C}]\boldsymbol{u}_j \tag{46}$$

The eigenvector derivative is given by Eq. (22). The vector $\boldsymbol{h}_j$ and the constant c_j can be obtained from Eqs. (21) and (28) as

$$\boldsymbol{h}_j = -\left[2\lambda_j\frac{\partial\lambda_j}{\partial p}\boldsymbol{M} + \lambda_j^2\frac{\partial \boldsymbol{M}}{\partial p} + \frac{\partial\lambda_j}{\partial p}\boldsymbol{C} + \lambda_j\frac{\partial \boldsymbol{C}}{\partial p} + \frac{\partial \boldsymbol{K}}{\partial p}\right]\boldsymbol{u}_j \tag{47}$$

$$c_j = -\frac{1}{\boldsymbol{u}_j^T\boldsymbol{D}'(\lambda_j)\boldsymbol{u}_j}\left\{\frac{1}{2}\boldsymbol{u}_j^T\left[2\frac{\partial\lambda_j}{\partial p}\boldsymbol{M} + 2\lambda_j\frac{\partial \boldsymbol{M}}{\partial p} + \frac{\partial \boldsymbol{C}}{\partial p}\right]\boldsymbol{u}_j + \boldsymbol{u}_j^T[2\lambda\boldsymbol{M} + \boldsymbol{C}]\boldsymbol{x}_j\right\} \tag{48}$$

C. Systems with Exponentially Decaying Damping Kernels

In this case the matrix of the damping kernel functions is expressed as

$$\boldsymbol{\mathcal{G}}(t) = \sum_{k=1}^{n}\mu_k e^{-\mu_k t}\boldsymbol{C}_k \tag{49}$$

or in the Laplace domain

$$\boldsymbol{G}(s) = \sum_{k=1}^{n}\frac{\mu_k}{s + \mu_k}\boldsymbol{C}_k \tag{50}$$

The constants $\mu_k \in \mathbb{R}^+$ are known as the relaxation parameters, and n denotes the number relaxation parameters used to describe the damping behavior. Several authors, for example, Muravyov and Hutton,[44] Muravyov,[45] Wagner and Adhikari,[25] and Adhikari and Wagner[26] have considered the dynamic analysis of such systems. From Eq. (6) observe that in the limit when $\mu_k \to \infty,\ \forall\, k$, the exponential model reduces to the viscous damping model with an equivalent viscous damping matrix

$$\boldsymbol{C} = \sum_{k=1}^{n}\boldsymbol{C}_k \tag{51}$$

Therefore the exponential kernel model is more general than the viscous damping model. Using $\boldsymbol{G}(s)$ in Eq. (50), it follows that

$$\boldsymbol{D}(s) = \left[s^2\boldsymbol{M} + s\sum_{k=1}^{n}\boldsymbol{C}_k\left(1+\frac{s}{\mu_k}\right)^{-1} + \boldsymbol{K}\right] \tag{52}$$

$$\boldsymbol{D}'(s) = \left[2s\boldsymbol{M} + \sum_{k=1}^{n}\boldsymbol{C}_k\left(1+\frac{s}{\mu_k}\right)^{-2}\right] \tag{53}$$

For this type of nonviscously damped system, the number of eigenvalues (see Refs. 25 and 26 for details) is $m = 2N + R$, where

$$R = \sum_{k=1}^{n}\operatorname{rank}(\boldsymbol{C}_k) \tag{54}$$

If all of the $\boldsymbol{C}_k$ matrices are of full rank, then $R = nN$. Among the m eigensolutions, $2N$ appear in complex conjugate pairs, and the other R eigensolutions are purely real. Using Eqs. (40) and (41), we obtain

$$\theta_j = \boldsymbol{u}_j^T\boldsymbol{D}'(\lambda_j)\boldsymbol{u}_j = \boldsymbol{u}_j^T\left[2s\boldsymbol{M} + \sum_{k=1}^{n}\boldsymbol{C}_k\left(1+\frac{s}{\mu_k}\right)^{-2}\right]\boldsymbol{u}_j \tag{55}$$

$$\left.\frac{\partial\boldsymbol{D}(s)}{\partial p}\right|_{s=\lambda_j} = \lambda_j^2\frac{\partial\boldsymbol{M}}{\partial p} + \lambda_j\sum_{k=1}^{n}\left[\frac{\partial\boldsymbol{C}_k}{\partial p}\left(1+\frac{\lambda_j}{\mu_k}\right)^{-1} + \frac{\lambda_j}{\mu_k^2}\boldsymbol{C}_k\frac{\partial\mu_k}{\partial p}\left(1+\frac{\lambda_j}{\mu_k}\right)^{-2}\right] + \frac{\partial\boldsymbol{K}}{\partial p} \tag{56}$$

$$\frac{\partial[\boldsymbol{D}(\lambda_j)]}{\partial p} = 2\lambda_j\frac{\partial\lambda_j}{\partial p}\boldsymbol{M} + \lambda_j^2\frac{\partial\boldsymbol{M}}{\partial p} + \frac{\partial\lambda_j}{\partial p}\boldsymbol{G}(\lambda_j) + \lambda_j\sum_{k=1}^{n}\left[\frac{\partial\boldsymbol{C}_k}{\partial p}\left(1+\frac{\lambda_j}{\mu_k}\right)^{-1} + \boldsymbol{C}_k\left(\frac{\lambda_j}{\mu_k^2}\frac{\partial\mu_k}{\partial p} - \frac{1}{\mu_k}\frac{\partial\lambda_j}{\partial p}\right)\left(1+\frac{\lambda_j}{\mu_k}\right)^{-2}\right] + \frac{\partial\boldsymbol{K}}{\partial p} \tag{57}$$

$$\frac{\partial[\boldsymbol{D}'(\lambda_j)]}{\partial p} = 2\frac{\partial\lambda_j}{\partial p}\boldsymbol{M} + 2\lambda_j\frac{\partial\boldsymbol{M}}{\partial p} + \sum_{k=1}^{n}\left[\frac{\partial\boldsymbol{C}_k}{\partial p}\left(1+\frac{\lambda_j}{\mu_k}\right)^{-2} + 2\boldsymbol{C}_k\left(\frac{\lambda_j}{\mu_k^2}\frac{\partial\mu_k}{\partial p} - \frac{1}{\mu_k}\frac{\partial\lambda_j}{\partial p}\right)\left(1+\frac{\lambda_j}{\mu_k}\right)^{-3}\right] \tag{58}$$

Using these expressions, the derivative of the eigenvalue can be obtained from Eq. (17). The eigenvector derivative is given by Eq. (22). The vector $\boldsymbol{h}_j$ and the constant c_j can be obtained using Eqs. (55–58).

The expressions in Eqs. (55–58) are quite general. In many practical application the damping coefficient matrices $\boldsymbol{C}_k$ can be independent of the relaxation parameter μ_k. In such cases either $\partial\boldsymbol{C}_k/\partial p$ or $\partial\mu_k/\partial p$ can be zero for a given design parameter p. This particular case will simplify the analytical expressions derived in Eqs. (56–58).

VI. Eigensolution Derivatives for the Non-Self-Adjoint Case

In this section we consider a more general case when $\boldsymbol{M}$, $\boldsymbol{\mathcal{G}}(t)$, and $\boldsymbol{K}$ are not restricted to symmetric matrices. For such asymmetric systems the adjoint eigenvalue problem or left eigenvalue problem is defined as

$$\boldsymbol{v}_j^T\left[\lambda_j^2\boldsymbol{M} + \lambda_j\boldsymbol{G}(\lambda_j) + \boldsymbol{K}\right] = 0 \qquad \text{or} \qquad \boldsymbol{v}_j^T\boldsymbol{D}(\lambda_j) = 0 \tag{59}$$

where $\boldsymbol{v}_j \in \mathbb{C}^N$ is the jth left eigenvector. Following the approach outlined in Ref. 17, Adhikari[36] obtained the derivative of eigenvalues of nonviscously damped asymmetric systems as

$$\frac{\partial\lambda_j}{\partial p} = -\frac{1}{\theta_j}\left[\boldsymbol{v}_j^T\left.\frac{\partial\boldsymbol{D}(s)}{\partial p}\right|_{s=\lambda_j}\boldsymbol{u}_j\right] \tag{60}$$

where the normalization constants are

$$\theta_j = \boldsymbol{v}_j^T\left.\frac{\partial\boldsymbol{D}(s)}{\partial s}\right|_{s=\lambda_j}\boldsymbol{u}_j \tag{61}$$

For the eigenvector derivatives, two problems arise in the non-self-adjoint case; the left and right eigenvector derivatives must be calculated simultaneously, and extra constraints must be introduced for the relative scaling of the left and right eigenvectors. The fact that the scaling given by Eq. (61) is insufficient to give unique eigenvectors can be demonstrated by multiplying the left eigenvector by any scalar and dividing the right eigenvector by the same scalar. The derivatives of the right eigenvectors are written as in Eq. (22), and the vector $\boldsymbol{x}_j$ is calculated as before. The derivatives of the left eigenvectors are written as

$$\frac{\partial\boldsymbol{v}_j}{\partial p} = \boldsymbol{y}_j + d_j\boldsymbol{v}_j \tag{62}$$

The vector $\boldsymbol{y}_j$ is obtained in a similar manner to $\boldsymbol{x}_j$. Equation (59) is differentiated, and Eq. (62) is used to obtain

$$\boldsymbol{y}_j^T\left[\lambda_j^2\boldsymbol{M} + \lambda_j\boldsymbol{G}(\lambda_j) + \boldsymbol{K}\right] = \boldsymbol{y}_j^T\boldsymbol{D}_j = \boldsymbol{g}_j \tag{63}$$

where

$$\boldsymbol{g}_j = -\boldsymbol{v}_j^T\frac{\partial\boldsymbol{D}(\lambda_j)}{\partial p} = -\boldsymbol{v}_j^T\left[2\lambda_j\frac{\partial\lambda_j}{\partial p}\boldsymbol{M} + \lambda_j^2\frac{\partial\boldsymbol{M}}{\partial p} + \frac{\partial\lambda_j}{\partial p}\boldsymbol{G}(\lambda_j) + \lambda_j\frac{\partial[\boldsymbol{G}(\lambda_j)]}{\partial p} + \frac{\partial\boldsymbol{K}}{\partial p}\right] \tag{64}$$

As for the right eigenvectors, the vector and scalar in Eq. (62) are not unique, but the same procedure of setting one of the elements of $\boldsymbol{v}_j$ to zero can be used.

It remains to compute the scalars c_j and d_j, using the eigenvector normalization. Differentiating Eq. (61) with respect to the parameter p and substituting the expressions for the eigenvector derivatives, Eqs. (22) and (62), produce

$$c_j + d_j = -\frac{1}{\theta_j}\left\{\boldsymbol{v}_j^T\frac{\partial\boldsymbol{D}'(\lambda_j)}{\partial p}\boldsymbol{u}_j + \boldsymbol{v}_j^T\boldsymbol{D}'(\lambda_j)\boldsymbol{x}_j + \boldsymbol{y}_j^T\boldsymbol{D}'(\lambda_j)\boldsymbol{u}_j\right\} \tag{65}$$

The matrix derivatives in Eq. (65) have already been discussed in detail for the self-adjoint case.

It remains to impose a constraint on the relative magnitudes of the eigenvectors. The best approach is to set one element in both eigenvectors to be equal. This element is arbitrary, but should be chosen so that this element has a large magnitude in both the left and the right eigenvectors. One possibility is to multiply the magnitudes of the corresponding elements of both eigenvectors and choose the largest product. Suppose that vector element number e is chosen. Then

$$\{\boldsymbol{u}_j\}_e = \{\boldsymbol{v}_j\}_e, \qquad \left\{\frac{\partial\boldsymbol{u}_j}{\partial p}\right\}_e = \left\{\frac{\partial\boldsymbol{v}_j}{\partial p}\right\}_e \tag{66}$$

This leads to a second simultaneous equation for c_j and d_j. If the same vector element number e is chosen for the normalization in

Table 1 Eigenvalues and eigenvectors for the example

Quantity	Elastic mode 1	Elastic mode 2	Nonviscous mode 1	Nonviscous mode 2
λ_j	$-0.0387 \pm 38.3232i$	$-1.5450 \pm 97.5639i$	-2.8403	-5.9923
$\boldsymbol{u}_j$	$\begin{Bmatrix} -0.7500 \pm 0.0043i \\ -0.6616 \mp 0.0041i \end{Bmatrix}$	$\begin{Bmatrix} 0.6622 \mp 0.0035i \\ -0.7501 \pm 0.0075i \end{Bmatrix}$	$\begin{Bmatrix} -0.0165 \\ 0.0083 \end{Bmatrix}$	$\begin{Bmatrix} 0.0055 \\ -0.0028 \end{Bmatrix}$

Table 2 Derivative of eigenvalues and eigenvectors with respect to the stiffness parameter k_1

Quantity	Elastic mode 1	Elastic mode 2	Nonviscous mode 1	Nonviscous mode 2
$\partial\lambda_j/\partial k_1$	$0.0001 \pm 0.0073i$	$0.0001 \pm 0.0022i$	-2.7106×10^{-4}	-2.9837×10^{-5}
$\partial\boldsymbol{u}_j/\partial k_1 \times 10^3$	$\begin{Bmatrix} 0.1130 \mp 0.0066i \\ 0.0169 \pm 0.0041i \end{Bmatrix}$	$\begin{Bmatrix} 0.0385 \mp 0.0015i \\ 0.0494 \mp 0.0026i \end{Bmatrix}$	$\begin{Bmatrix} 0.0072 \\ 0.0046 \end{Bmatrix}$	$\begin{Bmatrix} -0.0018 \\ -0.0018 \end{Bmatrix}$

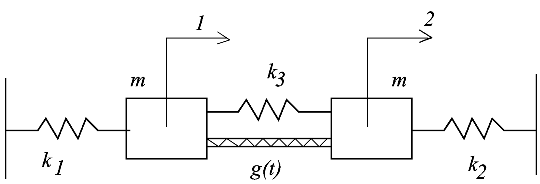

Fig. 1 Two-degree-of-freedom spring-mass system with nonviscous damping: $m = 1$ kg, $k_1 = 1000$ N/m, $k_2 = 2000$ N/m, $k_3 = 1600$ N/m, $g(t) = c(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t})$, $c = 200$ Ns/m, $\mu_1 = 5.0\,\mathrm{s}^{-1}$, and $\mu_2 = 7.0\,\mathrm{s}^{-1}$.

Eq. (66), and also as the zero element in $\boldsymbol{x}_j$ and $\boldsymbol{y}_j$, then Eq. (66) reduces to

$$c_j = d_j \tag{67}$$

which together with Eq. (65) yields the required solution for c_j and d_j.

VII. Numerical Example

We consider a two-degree-of-freedom system[36] shown in Fig. 1 to illustrate the use of the expressions derived in this paper.

Here the dissipative element connected between the two masses is not a simple viscous dashpot but a nonviscous damper. The equations of motion describing the free vibration of the system can be expressed by Eq. (1), with

$$\boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \qquad \boldsymbol{K} = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \tag{68}$$

$$\boldsymbol{\mathcal{G}}(t) = g(t)\hat{\boldsymbol{I}}, \qquad \text{where} \qquad \hat{\boldsymbol{I}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{69}$$

The damping function $g(t)$ is assumed to be a double exponential model, with

$$g(t) = c\left(\mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t}\right), \qquad c, \mu_1, \mu_2 \geq 0 \tag{70}$$

where c is a constant and μ_1 and μ_2 are known as the relaxation parameters. In Eq. (70) if the function associated with c was a delta function, c would be the familiar viscous damping constant. Taking the Laplace transform of Eq. (69), one obtains

$$\boldsymbol{G}(s) = c\hat{\boldsymbol{I}}\left\{(1 + s/\mu_1)^{-1} + (1 + s/\mu_2)^{-1}\right\} \tag{71}$$

Substituting Eqs. (68) and (71) into Eq. (6) shows that the system has six eigenvalues: four of which occur in complex conjugate pairs and correspond to the two elastic modes. The other two eigenvalues are real and negative, and they correspond to the two nonviscous modes. The eigenvalues and the eigenvectors of the system are shown in Table 1.

The normalization constants θ_j are selected such that $\theta_j = 2\lambda_j$ for the elastic modes and $\theta_j = 1$ for the nonviscous modes.

We consider the derivative of eigenvalues with respect to the stiffness parameter k_1 and the relaxation parameter μ_1. The derivative of the relevant system matrices with respect to k_1 can be obtained as

$$\frac{\partial \boldsymbol{M}}{\partial k_1} = \boldsymbol{O}, \quad \left.\frac{\partial \boldsymbol{G}(s)}{\partial k_1}\right|_{s=\lambda_j} = \boldsymbol{O}, \quad \frac{\partial \boldsymbol{K}}{\partial k_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{72}$$

$$\frac{\partial[\boldsymbol{D}(\lambda_j)]}{\partial k_1} = \left\{2\lambda_j \boldsymbol{M} + \boldsymbol{G}(\lambda_j) - c\lambda_j\hat{\boldsymbol{I}}\left[\mu_1^{-1}\left(1 + \frac{\lambda_j}{\mu_1}\right)^{-2} + \mu_2^{-1}\left(1 + \frac{\lambda_j}{\mu_2}\right)^{-2}\right]\right\}\frac{\partial \lambda_j}{\partial k_1} + \frac{\partial \boldsymbol{K}}{\partial k_1} \tag{73}$$

$$\frac{\partial[\boldsymbol{D}'(\lambda_j)]}{\partial k_1} = \left\{2\boldsymbol{M} - 2c\hat{\boldsymbol{I}}\left[\mu_1^{-1}\left(1 + \frac{\lambda_j}{\mu_1}\right)^{-3} + \mu_2^{-1}\left(1 + \frac{\lambda_j}{\mu_2}\right)^{-3}\right]\right\}\frac{\partial \lambda_j}{\partial k_1} \tag{74}$$

Using these expressions, the derivative of the eigenvalues and eigenvectors is obtained from Eqs. (17) and (22) and shown in Table 2.

The derivatives of the eigensolutions with respect to the relaxation parameter μ_1 can be obtained using similar manner. The derivative of the relevant system matrices with respect to μ_1 can be obtained as

$$\frac{\partial \boldsymbol{M}}{\partial \mu_1} = \boldsymbol{O}, \quad \frac{\partial \boldsymbol{K}}{\partial \mu_1} = \boldsymbol{O}, \quad \left.\frac{\partial \boldsymbol{G}(s)}{\partial \mu_1}\right|_{s=\lambda_j} = c\hat{\boldsymbol{I}}\,\lambda_j\mu_1^{-2}\left(1 + \frac{\lambda_j}{\mu_1}\right)^{-2} \tag{75}$$

$$\frac{\partial[\boldsymbol{D}(\lambda_j)]}{\partial \mu_1} = \left\{2\lambda_j \boldsymbol{M} + \boldsymbol{G}(\lambda_j) - c\lambda_j\hat{\boldsymbol{I}}\left[\mu_1^{-1}\left(1 + \frac{\lambda_j}{\mu_1}\right)^{-2} + \mu_2^{-1}\left(1 + \frac{\lambda_j}{\mu_2}\right)^{-2}\right]\right\}\frac{\partial \lambda_j}{\partial \mu_1} + c\hat{\boldsymbol{I}}\lambda_j^2\mu_1^{-2}\left(1 + \frac{\lambda_j}{\mu_1}\right)^{-2} \tag{76}$$

$$\frac{\partial[\boldsymbol{D}'(\lambda_j)]}{\partial \mu_1} = \left\{2\boldsymbol{M} - 2c\hat{\boldsymbol{I}}\left[\mu_1^{-1}\left(1 + \frac{\lambda_j}{\mu_1}\right)^{-3} + \mu_2^{-1}\left(1 + \frac{\lambda_j}{\mu_2}\right)^{-3}\right]\right\}\frac{\partial \lambda_j}{\partial \mu_1} + 2c\hat{\boldsymbol{I}}\,\lambda_j\mu_1^{-2}\left(1 + \frac{\lambda_j}{\mu_1}\right)^{-3} \tag{77}$$

Using these expressions, the derivative of the eigenvalues and eigenvectors is obtained from Eqs. (17) and (22) and shown in Table 3.

Table 3 Derivative of eigenvalues and eigenvectors with respect to the relaxation parameter μ_1

Quantity	Elastic mode 1	Elastic mode 2	Nonviscous mode 1	Nonviscous mode 2
$\partial\lambda_j/\partial\mu_1$	$-0.0034 \pm 0.0196i$	$-0.2279 \pm 2.0255i$	-0.0570	-0.4804
$\partial \boldsymbol{u}_j/\partial\mu_1 \times 10^3$	$\begin{Bmatrix} 0.0022 \pm 0.0004i \\ -0.0021 \mp 0.0003i \end{Bmatrix}$	$\begin{Bmatrix} -0.0045 \mp 0.0012i \\ 0.0098 \pm 0.0015i \end{Bmatrix}$	$\begin{Bmatrix} -0.0002 \\ 0.0001 \end{Bmatrix}$	$\begin{Bmatrix} 0.0022 \\ -0.0011 \end{Bmatrix}$

VIII. Conclusions

This paper has outlined a method to calculate the derivatives of eigenvalues and eigenvectors for systems with nonproportional and nonviscous damping. Nelson's method is used, which has the advantage that only the eigenvectors of interest are required. Undamped systems and nonproportionally damped viscous systems appear as special cases of this general derivation. For self-adjoint systems the usual eigenvector scaling is sufficient to obtain the eigenvector derivatives. For non-self-adjoint systems a further constraint is required to fix the relative magnitude of the left and right eigenvectors. Using this constraint the left and right eigenvector derivatives are calculated simultaneously. The method has been demonstrated on a simple example with an exponential damping kernel. However the same approach can be used for more complex models and for other models of nonviscous damping, such as the fractional derivative model. The derivatives can then be used for structural optimization, model updating, or other applications.

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