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Sensitivity based reduced approaches for structural reliability analysis

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Abstract. In the reliability analysis of a complex engineering structures a very large number of system parameters can be considered to be random variables. The difficulty in computing the failure probability increases rapidly with the number of variables. In this paper, a few methods are proposed whereby the number of variables can be reduced without compromising the accuracy of the reliability calculation. Based on the sensitivity of the failure surface, three new reduction methods, namely (a) gradient iteration method, (b) dominant gradient method, and (c) relative importance variable method, have been proposed. Numerical examples are provided to illustrate the proposed methods.

Keywords. Reliability analysis; optimization; approximation methods; FORM; SORM.

1. Introduction

Uncertainties in specifying material properties, geometric parameters, boundary conditions and applied loadings are unavoidable in describing real-life engineering structural systems. Traditionally, this has been catered for through the use of safety factors at the design stage. Such an approach may not be always satisfactory in today's competitive design environment, for example, in minimum weight design of aircraft structures. The situation may also arise when system safety is being jeopardized due to the lack of detailed treatment of uncertainty at the design stage, for example, finite probability of occurring a resonance is unlikely to be captured by a safety-factor based approach due to the intricate nonlinear relationships between the system parameters and the natural frequencies. For these reasons a scientific and systematic approach is required to predict the probability of failure of a structure at the design stage. Probabilistic structural reliability analysis is one such approach. This can be implemented in conjunction with the stochastic finite element, method (see for example Adhikari & Manohar 1999, 2000; Ghanem & Spanos 1991; Kleiber & Hien 1992; Manohar & Adhikari 1998a,b; Matthies *et al* 1997), to consider general structural systems. The books by Augusti *et al* (1984), Ditlevsen & Madsen (1996), Haldar & Mahadevan (2000), Melchers (1999), Thoft-Christensen & Baker (1982), Tichy (1993) and the review paper by Manohar & Gupta (2003) give the accounts of extensive research works which have been done over the last three decades.

The practical methods for reliability calculations can be broadly divided into (i) approximate analytical methods like FORM (First Order Reliability Method by Hasofer & Lind (1974)) and SORM (Second Order Reliability Method (Adhikari 2004; Cai & Elishakoff 1994; Der-Kiureghian *et al* 1987; Der-Kiureghian & Stefano 1991; Fiessler *et al* 1979; Hohenbichler & Rackwitz 1988; Hong 1999; Köylüoğlu & Nielsen 1994; Madsen *et al* 1986; Mahadevan & Shi 2001; Polidori *et al* 1999; Tvedt 1990; Zhao & Ono 1999a,b)); (ii) simulation based methods, for example, importance sampling (Bucher 1988; Mahadevan & Raghothamachar 2000; Schuëller & Stix 1987); (iii) methods based on surrogate modelling, for example response surface method (Bucher & Bourgund 1990; Faravelli 1989) and the improved response surface method (Gupta & Manohar 2004a,b); and (iv) artificial intelligence methods, such as neural networks (Chapman & Crossland 1995; Hurtado 2002) and genetic algorithms (Deng *et al* 2005; Shao & Murotsu 1999) based methods. There are several other methods which include, but not limited to, methods using intervening variables (Wang & Grandhi 1996), higher-order nonlinear approximations (Grandhi & Wang 1999), fast Fourier transformations (Penmetsa & Grandhi 2003) and other methods by Penmetsa & Grandhi (2002a,b). In each case the efficiency and applicability of a particular methodology largely depends on the efficient computation of the so called *design point*. The design point and the 'region' around it contains the most important information regarding the probability of failure of a structure. The calculation of the design point requires the solution of a constrained optimization problem. In the reliability analysis of a complex engineering structure a very large number of the system parameters can be considered to be random variables. The difficulty in computing the design point and consequently the failure probability increases rapidly with the number of variables. For gradient based optimization methods, it is often required to compute the gradient vector and the Hessian matrix numerically. If there are n random variables, then the computation of the gradient vector and the Hessian matrix require $(n+1)$ and $n(n+1)/2$ evaluations of the failure surface respectively. Thus, the computational expense is roughly proportional to $(n+1)+n(n+1)/2 = (n+1)(n+2)/2$. Clearly, for large n, computational expense can be significantly reduced if the number of random variables can be reduced. The aim of this paper is to consider methods whereby the number of variables can be reduced without compromising the accuracy of the solution.

The proposed reduction methods are based on the sensitivity of the failure surface in the transformed standard Gaussian space. If the failure surface is close to linear, then the design point obtained from these methods will be close to the exact design point obtained using the full set of random variables. However, if the failure surface is significantly nonlinear, the different reduction methods introduce different kind of errors. The nature of these errors are studied using a wide range of numerical examples. It is shown that the design point obtained using the proposed reduction methods have acceptable accuracy for many large scale structural engineering problems.

2. Brief review of approximate reliability analysis

Suppose the random variables describing the uncertainties of the structure and loading are considered to form a vector $\mathbf{y} \in \mathbb{R}^n$. The statistics of the system are fully described by the joint probability density function $p(\mathbf{y})$. In general the random variables $\mathbf{y}$ are non-Gaussian. In principle, these random variables can be transformed to a set of uncorrelated Gaussian random variables via the Rosenblatt transformation (Rosenblatt 1952). Further, they can be scaled so that each random variable has zero mean and unit variance. Suppose these

transformed and scaled random variables are $\mathbf{x} \in \mathbb{R}^n$ with a joint probability density function $p(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R} = (2\pi)^{-n/2} \exp(-\mathbf{x}^T\mathbf{x}/2)$. For a given set of variables $\mathbf{x}$ the structure will either fail under the applied (random) loading or will be safe. The condition of the structure for every $\mathbf{x}$ can be described by a safety margin $g(\mathbf{x}) : \mathbb{R}^n \mapsto \mathbb{R}$ such structure has failed if $g(\mathbf{x}) \leq 0$ and is safe if $g(\mathbf{x}) > 0$. Thus, the probability of failure is given by

$$P_f = \int_{g(\mathbf{x})\leq 0} p(\mathbf{x}) d\mathbf{x}. \tag{1}$$

The function $g(\mathbf{x})$ is also known as the limit-state function and the $(n-1)$-dimensional surface $g(\mathbf{x}) = 0$ is known as the failure surface. The central theme of a reliability analysis is to evaluate the multidimensional integral (1). For most real-life cases the dimensionality of the integral is large and consequently the exact evaluation of the integral in equation (1) is not possible. For this reason some kind of approximate method is required to evaluate this integral. Using the first-order reliability method (FORM) the probability of failure is given by

$$P_f = \Phi(-\beta) \quad \text{with} \quad \beta = (\mathbf{x}^{*^T}\mathbf{x}^*)^{1/2}, \tag{2}$$

where $\mathbf{x}^*$, the 'design point' is the solution of following optimization problem

$$\mathbf{x}^* = \min\{\mathbf{x}^T\mathbf{x}/2 \quad \text{subject to} \quad g(\mathbf{x}) \leq 0\}. \tag{3}$$

To obtain $\mathbf{x}^*$ construct the Lagrangian

$$\mathcal{L}(\mathbf{x}) = \mathbf{x}^T\mathbf{x}/2 + \lambda g(\mathbf{x}) \tag{4}$$

and set

$$\frac{\partial \mathcal{L}(\mathbf{x})}{\partial x_k} = 0 \quad \text{for} \quad k = 1, 2, \ldots, n. \tag{5}$$

Substituting $\mathcal{L}(\mathbf{x})$ into the above equation one obtains

$$\mathbf{x}^* = -\lambda \nabla g(\mathbf{x}^*). \tag{6}$$

Taking transpose and multiplying we have

$$(\mathbf{x}^{*^T}\mathbf{x}^*) = \lambda^2(\nabla g^T(\mathbf{x}^*)\nabla g(\mathbf{x}^*)). \tag{7}$$

From this, the Lagrange multiplier λ can be obtained as

$$\lambda = \frac{|\mathbf{x}^*|}{|\nabla g(\mathbf{x}^*)|}. \tag{8}$$

Hasofer & Lind (1974) defined the reliability index $\beta = |\mathbf{x}^*|$, which is the minimum distance of the failure surface from the origin. Substituting the value of λ from equation (8) into equation (6) one obtains

$$\mathbf{x}^* + |\mathbf{x}^*| \frac{\nabla g(\mathbf{x}^*)}{|\nabla g(\mathbf{x}^*)|} = 0 \tag{9}$$

$$\text{or} \quad \frac{\mathbf{x}^*}{|\mathbf{x}^*|} = -\frac{\nabla g(\mathbf{x}^*)}{|\nabla g(\mathbf{x}^*)|} = \boldsymbol{\alpha}^*, \tag{10}$$

where $\boldsymbol{\alpha}^*$ is the unit vector to the design point. This implies that at the design point the gradient vector to the failure surface and the vector from the origin are parallel. Once the reliability index or β is known, several more accurate reliability analyses, for example, second-order reliability methods (SORM) or importance sampling methods can be performed. Thus the calculation of the design point and the reliability index is crucial for all approximate reliability methods. Over the years several 'tailor-made' optimization algorithms have been developed to solve the optimization problem (3). Hasofer & Lind (1974) proposed an iterative method to obtain $\mathbf{x}^*$ from equation (3). Later, based on equation (9), Rackwitz & Fiessler (1978) proposed an efficient iterative method which is now known as RF algorithm. Several variants of the RF algorithm have been proposed and we refer the readers to Sudret & Der-Kiureghian (2000) for further discussions. In this paper, three methods are proposed for the calculation of $\mathbf{x}^*$ and β. Proposed methods are based on reduction of random variables using sensitivity based approach.

3. Method 1: Gradient iteration method

First-order sensitivity based approaches have been used widely (see Haukaas & Kiureghian 2005, 2006; Zona *et al* 2005, for some applications) in the context of risk and reliability analysis of complex engineering systems. For some point $\mathbf{x} \in \mathbb{R}^n$ the first-order Taylor series expansion of $g(\mathbf{x})$ about $\mathbf{x}^*$ can be given by

$$g(\mathbf{x}) \approx g(\mathbf{x}^*) + (\mathbf{x} - \mathbf{x}^*)^T \left.\frac{\partial g(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{X}=\mathbf{X}^*} \tag{11}$$

If $g(\mathbf{x})$ is linear then $\frac{\partial g(\mathbf{X})}{\partial \mathbf{X}}$ is independent of $\mathbf{x}$. In this case $\mathbf{x}^*$ will be simply the projection of the origin ($\mathbf{x} = \mathbf{0}$) on $g(\mathbf{x}) = 0$ (see Melchers 1999, Chapter 3). Also note that the outward normal vector to the hypersurface $g(\mathbf{x}) = 0$ is

$$c_i = \lambda \left.\frac{\partial g(\mathbf{x})}{\partial x_i}\right|_{\mathbf{x}=\mathbf{0}}. \tag{12}$$

Assume that $\boldsymbol{\nabla} g = \left\{\frac{\partial g(\mathbf{X})}{\partial x_i}\right\} \in \mathbb{R}^n$ is normalized so that $(\boldsymbol{\nabla} g^T \boldsymbol{\nabla} g) = 1$. This normalization is chosen for simplicity only and it is not a requirement for the applicability of the method to be proposed. For a truly linear $g(\mathbf{x})$ from equation (9) it can be easily seen that

$$\mathbf{x}^* = -\beta \boldsymbol{\nabla} g. \tag{13}$$

Motivated by this, for a general nonlinear limit-state function we express $\mathbf{x}$ by

$$\mathbf{x} = v \boldsymbol{\nabla} g, \tag{14}$$

where $v \in \mathbb{R}$ is a new random variable. In view of (14), the constrained optimization problem (3) becomes a simple search problem, that is we need to solve for v such that

$$\begin{aligned} &g(v\boldsymbol{\nabla} g) = 0 \\ \text{or} \quad &h_1(v) = 0, \quad \text{where} \quad h_1(\bullet) = g(\bullet \boldsymbol{\nabla} g) \end{aligned} \tag{15}$$

is a (nonlinear) function of a *single* variable v. Comparing (13) and (14) it is clear that $v = -\beta$. The zeros of $h_1(v)$ can be obtained easily since it is a function of a single variable.

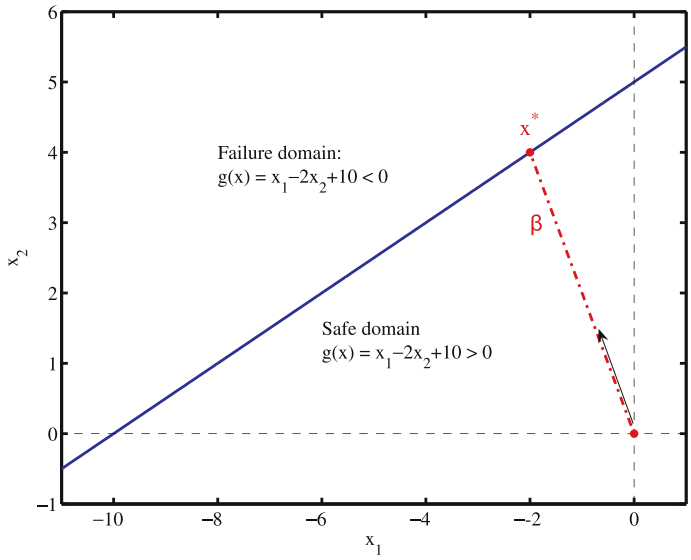

Figure 1. Linear failure surface in $\mathbb{R}^2$: $g(\mathbf{x}) \equiv g(x_1, x_2) = x_1 - 2x_2 + 10, \mathbf{x}^* = \{-2, 4\}$ and $\beta = 4{\cdot}472$.

Depending on the nature of nonlinearity of $h_1(v)$ it can have multiple real zeros. In this paper we exclude such cases. Indeed if $h_1(v) = 0$ has multiple real solutions, the critical value of v would be the smallest one since it would correspond to the highest probability of failure as $v = -\beta$. This method yields accurate result when $g(\mathbf{x})$ is linear or very close to linear. This is due to the fact that for linear $g(\mathbf{x})$, ∇g in the right-hand side of equation (14) is independent of the choice of $\mathbf{x}$ so that the direction of the outward normal from the failure surface does not change with position along the failure surface. For this reason $\nabla g|_{\mathbf{X}=\mathbf{0}}$ becomes the unit vector along true $\mathbf{x}^*$ and consequently β becomes simply the 'length' of this vector from the origin to the failure surface. To illustrate this approach consider the following example:

3.1 *Example 1: A linear failure surface with two variables*

Figure 1 shows a transformed failure surface in a two dimensional space $\mathbb{R}^2$.

The limit-sate function assumed to be linear with $g(\mathbf{x}) \equiv g(x_1, x_2) = x_1 - 2x_2 + 10$. It is trivial to show that for this simple example $\mathbf{x}^* = \{-2, 4\}^T$ and $\beta = 4{\cdot}472$. Now we will show how this result can be alternatively obtained using proposed approach.

First, we obtain the gradients of the failure surface:

$$\frac{\partial g(\mathbf{x})}{\partial x_1} = 1, \quad \frac{\partial g(\mathbf{x})}{\partial x_2} = -2. \tag{16}$$

Using these, the normalized gradient vector can be expressed as

$$\nabla g = \left\{ \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\}^T. \tag{17}$$

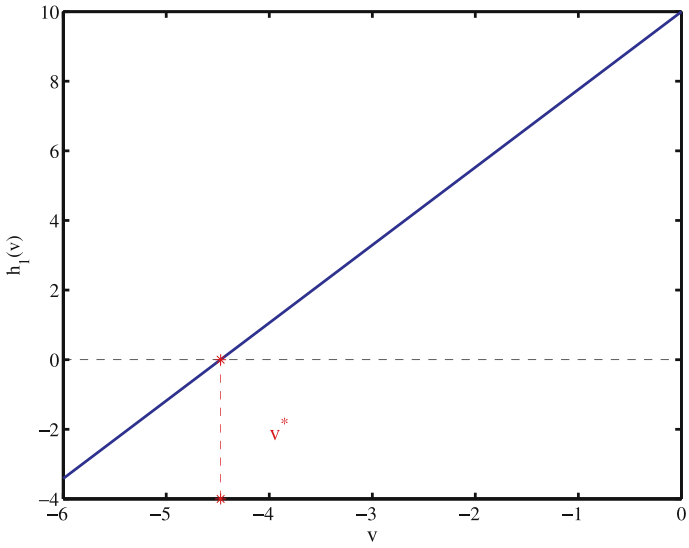

Figure 2. The function $h_1(v)$ and v^*, the solution $h_1(v) = 0$.

Since $\frac{\partial g(\mathbf{X})}{\partial x_i}$ are independent of $\mathbf{x}$, this is also the unit vector $\boldsymbol{\nabla} g|_{\mathbf{x}=\mathbf{0}}$ and is shown by the arrow in figure 1. Now we construct the function $h_1(v)$ as

$$h_1(v) = g(v\boldsymbol{\nabla} g) = g\left(\frac{v}{\sqrt{5}}, \frac{-2v}{\sqrt{5}}\right) = \frac{v + 4v}{\sqrt{5}} + 10. \tag{18}$$

The function $h_1(v)$ is plotted in figure 2 for different values of v. Clearly the solution of $h_1(v) = 0$ from equation (18) is $v^* = -2\sqrt{5} = -4{\cdot}472$. Therefore, $\beta = -v^* = 4{\cdot}472$ and the design point $\mathbf{x}^* = v^*\boldsymbol{\nabla} g = -2\sqrt{5}\{\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\}^T = \{-2, 4\}^T$, which are exactly the same as obtained using full set of two random variables.

The simple facts arising for a linear failure surface as shown in this example do not hold when $g(\mathbf{x})$ is a nonlinear function. In this case $\boldsymbol{\nabla} g$ depends on the choice of $\mathbf{x}$ and the direction of the outward normal from the failure surface changes along the failure surface. Therefore, $\boldsymbol{\nabla} g|_{\mathbf{x}=\mathbf{0}}$ is in general not the unit vector along true $\mathbf{x}^*$. To solve this problem we propose an iterative method so that $\boldsymbol{\nabla} g$ is updated at each iteration step. We first obtain an initial $\mathbf{x}^*$, say $\mathbf{x}_0^*$, by projecting $\boldsymbol{\nabla} g|_{\mathbf{x}=\mathbf{0}}$ to the failure surface. Next we use this point to obtain a new unit vector $\boldsymbol{\nabla} g|_{\mathbf{x}=\mathbf{x}_0^*}$. Projecting this vector from the origin to the failure surface we obtain the next estimate of $\mathbf{x}^*$, say $\mathbf{x}_1^*$. The method then uses this point to calculate $\boldsymbol{\nabla} g$ and continues until two successive estimates of design points are close enough. In summary, for a general smooth differentiable nonlinear $g(\mathbf{x})$, the iterative procedure can be described as follows:

1. For $k = 0$, select $\mathbf{x}^{(0)} = \mathbf{0}$, a small value of ϵ, say $\epsilon = 0{\cdot}001$, a large value of $\beta^{(0)}$, say $\beta^{(0)} = 10$.
2. Calculate $\nabla g_i^{(k)} = \frac{\partial g(\mathbf{X})}{\partial x_i}\big|_{\mathbf{X}=\mathbf{X}^{(k)}}$ for $i = 1, \ldots, n$. Construct the vector $\boldsymbol{\nabla} g^{(k)} = \{\nabla g_i^{(k)}\} \in \mathbb{R}^n$ and normalize so that $(\boldsymbol{\nabla} g^{(k)^T} \boldsymbol{\nabla} g^{(k)}) = 1$.
3. Use the transformation $\mathbf{x} = v\boldsymbol{\nabla} g^{(k)}$ to obtain $h_1(v)$ from $g(\mathbf{x})$.
4. Solve $h_1(v) = 0$ for v.

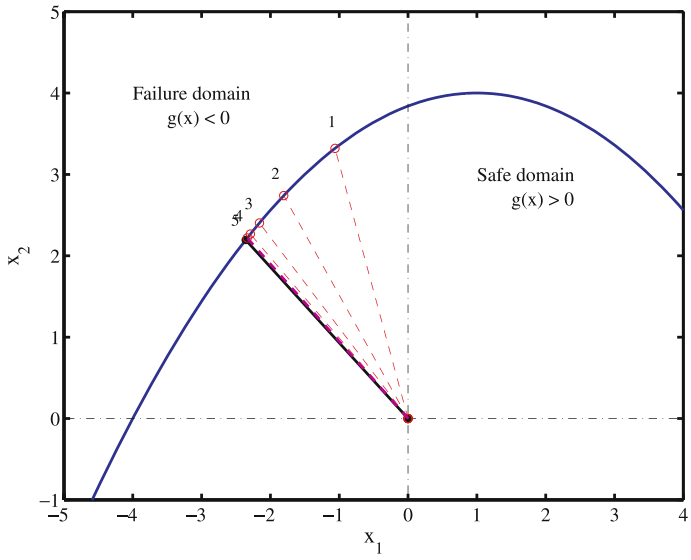

Figure 3. Nonlinear failure surface in $\mathbb{R}^2$: $g(\mathbf{x}) = -\frac{4}{25}(x_1 - 1)^2 - x_2 + 4 = 0$; '—' actual design vector $\mathbf{x}^*$ obtained using full set of variables; '– –' design vectors obtained at each iteration step of the proposed iterative method.

5. Increase the index: $k = k + 1$; denote $\beta^{(k)} = -v$ and $\mathbf{x}^{(k)} = v\boldsymbol{\nabla} g^{(k)}$.
6. Denote $\delta\beta = \beta^{(k-1)} - \beta^{(k)}$.
7. (a) If $\delta\beta < 0$ then the iteration is going in the wrong direction. Terminate the iteration procedure and select $\beta = \beta^{(k)}$ and $\mathbf{x}^* = \mathbf{x}^{(k)}$ as the best values of these quantities.
   (b) If $\delta\beta < \epsilon$ then the iterative procedure has converged. Terminate the iteration procedure and select $\beta = \beta^{(k)}$ and $\mathbf{x}^* = \mathbf{x}^{(k)}$ as the final values of these quantities.
   (c) If $\delta\beta > \epsilon$ then go back to step 2.

It should be noted that the convergence of the proposed iterative method cannot be always guaranteed. It is hoped that if the failure surface is fairly regular and smooth then the method would converge. Next, this method is illustrated through numerical examples.

3.2 *Example 2: Nonlinear failure surface with two variables:*

Consider a system with two random variables x_1 and x_2. The nonlinear failure surface is given by

$$g(\mathbf{x}) = -\frac{4}{25}(x_1 - 1)^2 - x_2 + 4 = 0. \tag{19}$$

This example is taken from Melchers (1999) (page 105). Figure 3 shows the failure surface together with the design vector. The actual design vector and reliability index obtained by Melchers (1999) is given by

$$\mathbf{x}^* = \{-2{\cdot}36, 2{\cdot}19\}^T \quad \text{and} \quad \beta = 3{\cdot}22. \tag{20}$$

Table 1. Numerical values of the design vector and reliability index at each iteration step of the proposed iterative method.

Iteration number	$\mathbf{x}^*$	β
1	$\{-1{\cdot}0623, 3{\cdot}3195\}^T$	3·4854
2	$\{-1{\cdot}8075, 2{\cdot}7389\}^T$	3·2815
3	$\{-2{\cdot}1591, 2{\cdot}4033\}^T$	3·2307
4	$\{-2{\cdot}2914, 2{\cdot}2667\}^T$	3·2231
5	$\{-2{\cdot}3367, 2{\cdot}2186\}^T$	3·2222

To apply the proposed method we need to obtain the gradients of the failure surface:

$$\frac{\partial g(\mathbf{x})}{\partial x_1} = -\frac{8}{25}(x_1 - 1), \quad \frac{\partial g(\mathbf{x})}{\partial x_2} = -1. \tag{21}$$

The starting point of the iterative scheme is $\mathbf{x} = \{0, 0\}^T$. Table 1 shows the values of β and $\mathbf{x}^*$ at each iteration step using the proposed method.

Figure 4 shows the function $h_1(v)$ at each iteration step. Note that after five iterations the results obtained from the proposed method converge to the actual values.

3.3 *Example 3: Nonlinear failure surface with three variables:*

Consider a system with three random variables so that $\mathbf{x} = \{x_1, x_2, x_3\}^T$. The nonlinear failure surface is given by

$$g(\mathbf{x}) = -\frac{4}{25}(x_1 + 1)^2 - \frac{(x_2 - 5/2)^2(x_1 - 5)}{10} - x_3 + 3. \tag{22}$$

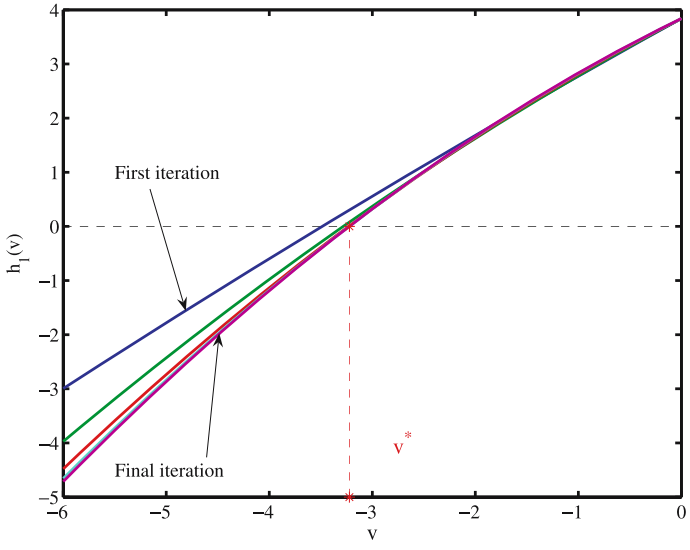

Figure 4. The function $h_1(v)$ and v^*, the solution $h_1(v) = 0$ for different iteration steps for the failure surface as in figure 3.

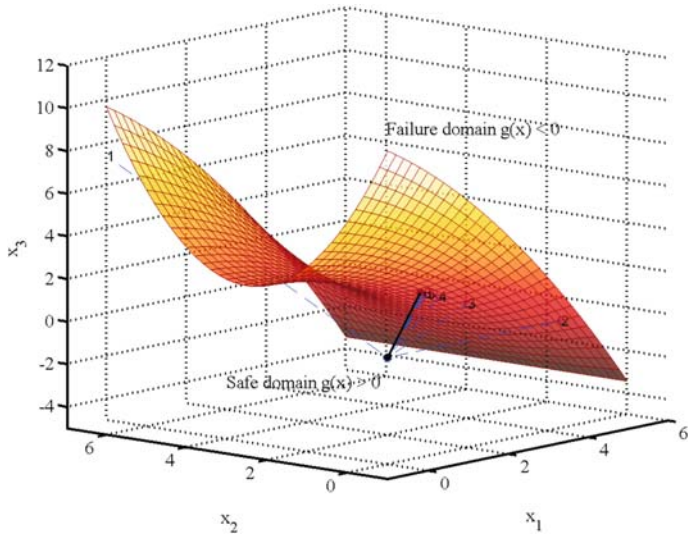

Figure 5. Nonlinear failure surface in $\mathbb{R}^3$: $g(\mathbf{x}) = -\frac{4}{25}(x_1+1)^2 - \frac{(x_2-5/2)^2(x_1-5)}{10} - x_3 + 3$; '—' actual design vector $\mathbf{x}^*$ obtained using full set of variables; '– –' design vectors obtained at each iteration step of the proposed iterative method.

Figure 5 shows the failure surface together with the design vector. The actual design vector and reliability index obtained by using all the three random variables is given by

$$\mathbf{x}^* = \{2{\cdot}1286,\ 1{\cdot}2895,\ 1{\cdot}8547\}^T \quad \text{and} \quad \beta = 3{\cdot}1038. \tag{23}$$

To apply the proposed method, we need to obtain the gradients of the failure surface:

$$\frac{\partial g(\mathbf{x})}{\partial x_1} = -\frac{8}{25}(x_1+1) - \frac{(x_2-5/2)^2}{10}, \quad \frac{\partial g(\mathbf{x})}{\partial x_2} = -\frac{(x_2-5/2)(x_1-5)}{5},$$
$$\frac{\partial g(\mathbf{x})}{\partial x_3} = -1. \tag{24}$$

The starting point of the iterative scheme is $\mathbf{x} = \{0, 0, 0\}^T$. Table 2 shows the values of β and $\mathbf{x}^*$ at each iteration step using the proposed method.

Table 2. Numerical values of the design vector and reliability index at each iteration step of the proposed iterative method.

Iteration number	$\mathbf{x}^*$	β
1	$\mathbf{x}^* = \{4{\cdot}2245,\ 11{\cdot}1758,\ 4{\cdot}4703\}^T$	12·7565
2	$\mathbf{x}^* = \{3{\cdot}8085,\ -0{\cdot}5572,\ 0{\cdot}4140\}^T$	3·8713
3	$\mathbf{x}^* = \{2{\cdot}8825,\ 0{\cdot}8490,\ 1{\cdot}1654\}^T$	3·2230
4	$\mathbf{x}^* = \{2{\cdot}4309,\ 1{\cdot}1219,\ 1{\cdot}6046\}^T$	3·1213
5	$\mathbf{x}^* = \{2{\cdot}2506,\ 1{\cdot}2375,\ 1{\cdot}7476\}^T$	3·1066
6	$\mathbf{x}^* = \{2{\cdot}1788,\ 1{\cdot}2610,\ 1{\cdot}8163\}^T$	3·1042

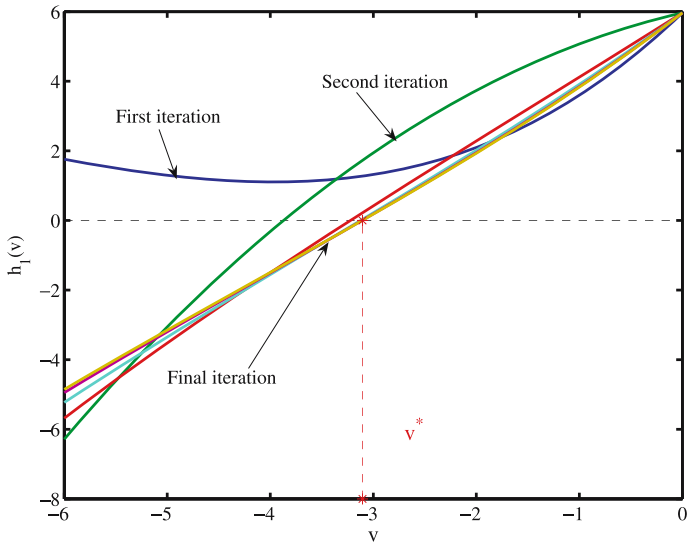

Figure 6. The function $h_1(v)$ and v^*, the solution $h_1(v) = 0$ for different iteration steps for the failure surface as in figure 5.

Figure 6 shows the function $h_1(v)$ at each iteration step. Note that after six iterations the results obtained from the proposed method converge to the actual values.

3.4 *Example 4: Non-smooth failure surface with two variables:*

We consider a failure surface similar to that in example 2, except that in this time it has been perturbed by a sinusoidal noise. The assumed non-smooth failure surface is given by

$$g(\mathbf{x}) = -\frac{4}{25}(x_1 - 1)^2 - x_2 + 4 + \frac{1}{5}\sin(5x_1) = 0. \tag{25}$$

Figure 7 shows the failure surface together with the design vector. The actual design vector and reliability index obtained using full set of variables can be obtained as

$$\mathbf{x}^* = \{-1{\cdot}619, 2{\cdot}708\}^T \quad \text{and} \quad \beta = 3{\cdot}155. \tag{26}$$

The design point for each iteration steps is shown in figure 7. After 16 iterations the method converges to:

$$\mathbf{x}^* = \{-2{\cdot}732, 1{\cdot}5932\}^T \quad \text{and} \quad \beta = 3{\cdot}163. \tag{27}$$

Note that although the value of β is close to the actual value, the design point obtained using the proposed method is quite far from the actual design point. This example shows that for non-smooth failure surfaces the proposed method may converge to a wrong design point.

4. Method 2: Dominant gradient method

The reduction method outlined in the previous section effectively uses only one variable. Here we keep more than one random variable in the analysis. The criteria for selecting the

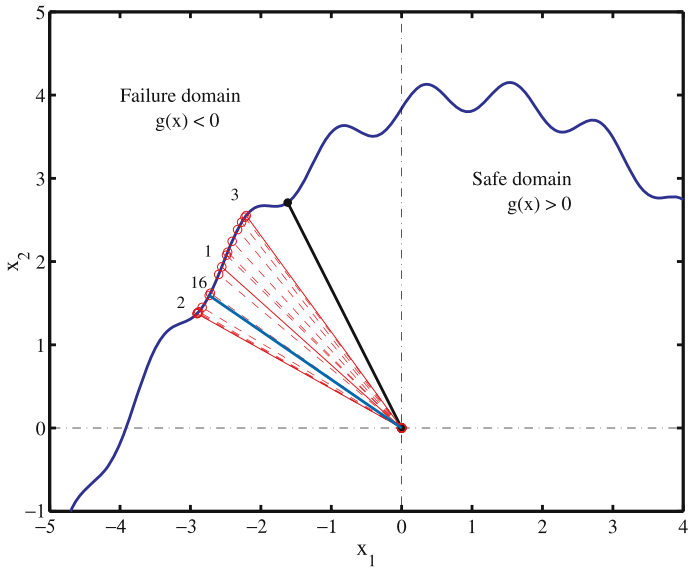

Figure 7. Non-smooth failure surface in $\mathbb{R}^2$: $g(\mathbf{x}) = -\frac{4}{25}(x_1 - 1)^2 - x_2 + 4 + \frac{1}{5}\sin(5x_1) = 0$; '—' actual design vector $\mathbf{x}^*$ obtained using full set of variables; '– –' design vectors obtained at each iteration step of the proposed iterative method.

random variables is based on the sensitivity of the failure surface with respect to the variables. Assume that

$$\boldsymbol{\nabla} g = \left\{ \left. \frac{\partial g(\mathbf{x})}{\partial x_i} \right|_{\mathbf{x}=\mathbf{0}} \right\} \in \mathbb{R}^n \tag{28}$$

is normalized so that $(\boldsymbol{\nabla} g^T \boldsymbol{\nabla} g) = 1$. Suppose only n_d entries of $\boldsymbol{\nabla} g$ has significant non-zero (negative or positive) values while all the other entries are close to zero. Consider the index set of these dominant variables is given by I_d. We construct the vector of dominant random variables $\mathbf{v} \in \mathbb{R}^{n_d}$ so that

$$v_j = x_i, \quad \forall j = 1, \ldots, n_d \text{ and } i \in I_d. \tag{29}$$

This equation can be represented in a matrix form as

$$\mathbf{v} = \mathbf{D}\mathbf{x}, \tag{30}$$

where $\mathbf{D} \in \mathbb{R}^{n_d \times n}$ is such that $\mathbf{D}_{ji} = 1 \forall j = 1, \ldots, n_d; i \in I_d$ and zero everywhere else. All other random variables are assumed to be zero, that is $x_i = 0, \forall i \notin I_d$. This implies that these variables assumed to be deterministic so that they do not play any role in the reliability analysis (see the discussion on 'omission sensitivity factors' by Madsen 1988). Using these reduced set of variables one may obtain the failure surface in the reduced space $h_2(\mathbf{v})$. Thus the optimization problem in the reduced space reads

$$\min \quad (\mathbf{v}^T \mathbf{v}/2) \quad \text{subject to} \quad h_2(\mathbf{v}) = 0. \tag{31}$$

The design point in the reduced space, $\mathbf{v}^*$, can be obtained from the solution of (31). From $\mathbf{v}^*$, the design point in the actual space can simply be obtained by substituting $x_i^* = v_j^*, \forall j = 1, \ldots, n_d; i \in I_d$ and $x_i^* = 0, \forall i \notin I_d$.

It should be noted that this approach is based on the sensitivity of $g(\mathbf{x})$ at $\mathbf{x} = \mathbf{0}$. For linear failure surface $\boldsymbol{\nabla} g|_{\mathbf{X}=\mathbf{0}}$ is the unit vector along the true $\mathbf{x}^*$ and the dominant random variables selected by the above procedure are actually the dominant random variables at the design point. For nonlinear $g(\mathbf{x})$, $\boldsymbol{\nabla} g|_{\mathbf{X}=\mathbf{0}}$ is in general not the unit vector along true $\mathbf{x}^*$. Thus the dominant random variables selected from $\boldsymbol{\nabla} g|_{\mathbf{X}=\mathbf{0}}$ are not necessarily the dominant random variables at the design point. Ideally the dominant random variables should be selected on the basis of $\boldsymbol{\nabla} g$ evaluated at the design point. Keeping this in mind, we update $\boldsymbol{\nabla} g$ in an iterative way so that the dominant random variables are selected on the basis of $\boldsymbol{\nabla} g$ evaluated at points gradually closer to the true design point. First, based on $\boldsymbol{\nabla} g|_{\mathbf{X}=\mathbf{0}}$, an initial $\mathbf{v}^*$ is obtained by solving the optimization problem (31). From this initial $\mathbf{v}^*$ the initial design point in the original space, say $\mathbf{x}_0^*$, is obtained. Next we use this point to obtain a new sensitivity vector $\boldsymbol{\nabla} g|_{\mathbf{X}=\mathbf{X}_0^*}$. Based on this we again select the dominant random variables and repeat the procedure to obtain the next estimate of $\mathbf{x}^*$, say $\mathbf{x}_1^*$. The method then uses this point to calculate $\boldsymbol{\nabla} g$ and continues until two successive estimates of design points are close enough. In summary, for nonlinear $g(\mathbf{x})$, the iterative procedure can be described as follows:

1. For $k = 0$, select $\mathbf{x}^{(0)} = \mathbf{0}$, a small value of ϵ, say $\epsilon = 0{\cdot}001$, a large value of $\beta^{(0)}$, say $\beta^{(0)} = 10$ and also $n_d < n$.
2. Calculate $\nabla g_i^{(k)} = \frac{\partial g(\mathbf{X})}{\partial x_i}\big|_{\mathbf{X}=\mathbf{X}^{(k)}}$. For $i = 1, \ldots, n$ construct the vector $\boldsymbol{\nabla} g^{(k)} = \{\nabla g_i^{(k)}\} \in \mathbb{R}^n$ and normalize so that $(\boldsymbol{\nabla} g^{(k)^T} \boldsymbol{\nabla} g^{(k)}) = 1$.
3. Sort $|\nabla g_i^{(k)}|$ to obtain the index set I_d corresponding to the highest n_d values.
4. Set $v_j = x_i^{(k)}, \forall j = 1, \ldots, n_d, i \in I_d$ and $x_i^{(k)} = 0, \forall i \notin I_d$. Construct $\mathbf{v} = \{v_j\} \in \mathbb{R}^{n_d}$.
5. Using this transformation obtain $h_2(\mathbf{v})$ from $g(\mathbf{x})$.
6. Solve the constrained optimization problem: minimize $\beta = (\mathbf{v}^T \mathbf{v})^{1/2}$ subject to $h_2(\mathbf{v}) = 0$.
7. Increase the index: $k = k + 1$. Using the solutions from step 6 denote $\beta^{(k)} = \beta$ and $\mathbf{v}^{(k)} = \mathbf{v}$.
8. Obtain $\mathbf{x}^{(k)}$ from the inverse transformation in step 4, that is $x_i^{(k)} = v_j, \forall j = 1, \ldots, n_d$, $i \in I_d$ and $x_i^{(k)} = 0, \forall i \notin I_d$.
9. Denote $\delta\beta = \beta^{(k-1)} - \beta^{(k)}$.
10. (a) If $\delta\beta < 0$ then the iteration is going in the wrong direction. Terminate the iteration procedure and select $\beta = \beta^{(k)}$ and $\mathbf{x}^* = \mathbf{x}^{(k)}$ as the best values of these quantities.
    (b) If $\delta\beta < \epsilon$ then the iterative procedure has converged. Terminate the iteration procedure and select $\beta = \beta^{(k)}$ and $\mathbf{x}^* = \mathbf{x}^{(k)}$ as the final values of these quantities.
    (c) If $\delta\beta > \epsilon$ then go back to step 2.

The initial choice of the dominant random variables, that is, n_d and I_d, can be automated by imposing a selection criteria, for example, by fixing the ratio of $\nabla g_i^{(k)}$ corresponding to the most sensitive random variable and the least sensitive random variable. Note that the index set of the dominant variables I_d may change in different iterations, however we fix n_d for all iterations. Like the method in section 3, the convergence of the proposed iterative method cannot be always guaranteed. In the next section another variant of this approach, which considers the contribution of the neglected variables, is proposed.

5. Method 3: Relative importance variable method

Based on the entries of $\boldsymbol{\nabla} g$, we group the random variables into the 'important' and 'unimportant' random variables. The random variables for which the failure surface is more sensitive

are called important variables. Like the approach in section 4, suppose only $n_d < n$ entries of ∇g with an index set $i \in I_d$ is important. Suppose that the important random variables are casted in the vector $\mathbf{v} \in \mathbb{R}^{n_d}$ such that equation (29) is satisfied. However, unlike the approach in section 4, we do not completely neglect all the 'unimportant' random variables, but consider that they can be represented by a single random variable, say u such that

$$x_i = u \nabla g_i, \quad \forall i \notin I_d. \tag{32}$$

This equation implies that the 'direction' of the unimportant random variables are fixed according to the gradient vector so that u is effectively a scaling parameter in that direction. This method can be viewed as a combination of the methods described in the previous two sections. Now we construct the vector of reduced random variables $\mathbf{z}$ as

$$\mathbf{z} = \{\mathbf{v}, u\}^T \in \mathbb{R}^{n_d+1}. \tag{33}$$

Using equations (29) and (32) one may obtain the failure surface in the reduced space $h_3(\mathbf{z})$. Thus the optimization problem in the reduced space reads

$$\min \quad (\mathbf{z}^T \mathbf{z})^{1/2} \quad \text{subject to} \quad h_3(\mathbf{z}) = 0. \tag{34}$$

The design point in the reduced space, $\mathbf{z}^* = \{\mathbf{v}^*, u^*\}^T$, can be obtained from the solution of (34). From $\mathbf{z}^*$, the design point in the actual space can simply be obtained by substituting $x_i^* = v_j^*, \forall j = 1, \ldots, n_d; i \in I_d$ and $x_i^* = u \nabla g_i, \forall i \notin I_d$.

The selection of the important and unimportant random variables are again based on ∇g evaluated at $\mathbf{x} = \mathbf{0}$. The important and unimportant random variables should be selected on the basis of ∇g evaluated at the true design point $\mathbf{x}^*$. Because for nonlinear $g(\mathbf{x})$, $\nabla g|_{\mathbf{x}=\mathbf{0}}$ is in general not the unit vector along true $\mathbf{x}^*$, we update ∇g by an iterative method similar to the ones described in the previous two sections. In summary this iterative approach can be described as follows:

1. For $k = 0$, select $\mathbf{x}^{(0)} = \mathbf{0}$, a small value of ϵ, say $\epsilon = 0{\cdot}001$, a large value of $\beta^{(0)}$, say $\beta^{(0)} = 10$ and also $n_d < n$.
2. Calculate $\nabla g_i^{(k)} = \frac{\partial g(\mathbf{X})}{\partial x_i}\big|_{\mathbf{X}=\mathbf{X}^{(k)}}$. For $i = 1, \ldots, n$ construct the vector $\nabla g^{(k)} = \{\nabla g_i^{(k)}\} \in \mathbb{R}^n$ and normalize so that $\nabla g^{(k)^T} \nabla g^{(k)} = 1$.
3. Sort $|\nabla g_i^{(k)}|$ to obtain the index set I_d corresponding to the highest n_d values.
4. Set $v_j = x_i^{(k)}, \forall j = 1, \ldots, n_d$, $i \in I_d$ and $x_i^{(k)} = u \nabla g_i^{(k)}, \forall i \notin I_d$. Construct $\mathbf{z} = \{v_j, u\} \in \mathbb{R}^{n_d+1}$.
5. Using this transformation obtain $h_3(\mathbf{z})$ from $g(\mathbf{x})$.
6. Solve the constrained optimization problem: minimize $\beta = (\mathbf{z}^T \mathbf{z})^{1/2}$ subject to $h_3(\mathbf{z}) = 0$.
7. Increase the index: $k = k + 1$. Using the solutions from step 6 denote $\beta^{(k)} = \beta$ and $\mathbf{z}^{(k)} = \mathbf{z}$.
8. Obtain $\mathbf{x}^{(k)}$ from the inverse transformation in step 4.
9. Denote $\delta\beta = \beta^{(k-1)} - \beta^{(k)}$.
10. (a) If $\delta\beta < 0$ then the iteration is going in the wrong direction. Terminate the iteration procedure and select $\beta = \beta^{(k)}$ and $\mathbf{x}^* = \mathbf{x}^{(k)}$ as the best values of these quantities.
    (b) If $\delta\beta < \epsilon$ then the iterative procedure has converged. Terminate the iteration procedure and select $\beta = \beta^{(k)}$ and $\mathbf{x}^* = \mathbf{x}^{(k)}$ as the final values of these quantities.
    (c) If $\delta\beta > \epsilon$ then go back to step 2.

Again, like the methods in sections 3 and 4, the convergence of the proposed iterative method cannot be always guaranteed. In the next section, the proposed methods are applied to some structural engineering problems.

6. Discussions on the proposed methods

The three methods proposed in the previous sections are aimed at handling failure surfaces with progressively increasing nonlinearity. As shown in section 3, if the failure surface is linear then only one variable is enough to obtain the design point exactly. But as the failure becomes more and more nonlinear, more variables are needed to obtain the design point efficiently. This is due to the fact that the gradient of a nonlinear failure surface is not constant with respective to the variables. The methods proposed in the paper exploits this fact. If the failure surface is fairly linear then the gradient iteration method (method 1) can be used. For increasingly nonlinear failure surfaces, dominant gradient method and relative importance variable method (methods 2 and 3) should be used respectively.

Once the reliability index is obtained using one of the proposed reduced methods, several tools are available to obtain the failure probability accurately. They include, but not limited to, second-order reliability methods (SORM), response surface based methods and importance sampling based methods. Note that for importance sampling based methods or response surface based method, the design point does not have to be very accurate. Therefore, one can terminate the proposed iterative methods at an early stage to accept a less accurate result. This can be easily achieved by increasing the value of $\delta\beta$ in the iterative algorithms.

For the case when the basic random variables are non-Gaussian, the failure surface can be transformed into the standard normal space using the Rosenblatt transformation (Rosenblatt 1952). The methods developed in the paper can be in principle applied to this transformed surface. However, it is well known that even for simple original failure surface, the transformed failure surface can be significantly nonlinear and complex. Future research is needed to investigate the applicability of the proposed method for such situations.

The design point loses its significance with the increase in dimension and nonlinearity. Hence, for $n > 30$ and considerable nonlinearities, the design point based predictions become increasingly inaccurate. Within the past few years several powerful methods have been proposed, for example, Koutsourelakis (2004), Koutsourelakis *et al* (2004), Schuëller *et al* (2004), for systems with large number of random variables. Using an asymptotic approach for $n \to \infty$, Adhikari (2004, 2005) proved that

$$P_f \to \Phi\left(-\frac{\beta + \text{Trace}(\mathbf{A})}{\sqrt{1 + 2\text{Trace}(\mathbf{A}^2)}}\right), \quad \text{when} \quad n \to \infty. \tag{35}$$

Here $\mathbf{A} \in \mathbb{R}^{(n-1)\times(n-1)}$ is related to the matrix of the curvatures (the Hessian matrix) at the design point. Equation (35) can be viewed as the correction needed to the FORM expression (2) when the number of random variables become large. Equation (35) is asymptotically correct and numerical results confirm that it becomes very accurate when n is large (equation (35) however becomes invalid for small values of n). The methods proposed in the previous sections can be used to obtain the design point, which in turn can be utilized in equation (35) to obtain the probability of failure. Because the difference between FORM and other methods are well established and the focus of this paper is on the derivation of β, only FORM is used in the numerical works to be followed in the next section.

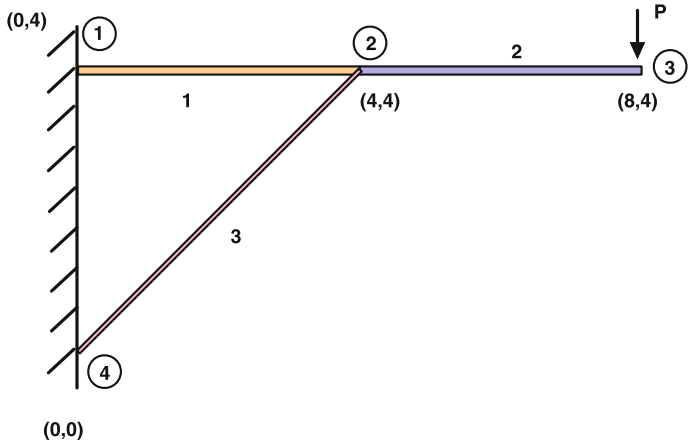

Figure 8. 2D frame with random element properties, $P = 100$ KN.

7. Numerical examples

7.1 *2D Frame structure*

We consider a 2D frame structure with three members. The structure is shown in figure 8 with element numbering, node numbering and coordinates of the nodes in meters. It is assumed that the axial stiffness (EA) and the bending stiffness (EI) of each member are Gaussian random variables so that there are in total six random variables, $\mathbf{x} \in \mathbb{R}^6$. Further, it is also assumed that EA and EI of different members are uncorrelated, that is

$$\langle EA_i, EA_j \rangle = 0, \forall i \neq j; \ \langle EI_i, EI_j \rangle = 0, \forall i \neq j; \ \langle EA_i, EI_j \rangle = 0, \forall i, j. \tag{36}$$

Therefore, the joint probability density function (pdf) of the random variables is completely characterized by the mean and the standard deviation of the random variables. Table 3 shows the numerical values of the system properties assumed for different members.

The standard deviations are expressed as a percentage of the corresponding mean values. The vertical force applied in node 3 is assumed to be 100 KN. The failure condition is given by specifying a maximum allowable vertical displacement at node 3, say $d_{\max}$. The failure

Table 3. Element properties of the random 2D frame as in figure 8. The standard deviations are expressed as a percentage of the corresponding mean values.

	EA (KN)		EI (KNm^2)	
Member Id	Mean	Standard deviation	Mean	Standard deviation
1	$1{\cdot}0 \times 10^9$	3·0%	$2{\cdot}0 \times 10^4$	10·0%
2	$5{\cdot}0 \times 10^9$	7·0%	$6{\cdot}0 \times 10^4$	5·0%
3	$3{\cdot}0 \times 10^9$	10·0%	$4{\cdot}0 \times 10^4$	9·0%

Table 4. Comparison of reliability index and failure probability of the random 2D frame as in figure 8.

Method	β	P_f
Gradient Iteration Method (four iterations)	3.590	$0{\cdot}165 \times 10^{-3}$
Dominant Gradient Method with $n_d = 3$ (one iteration)	3·590	$0{\cdot}165 \times 10^{-3}$
Relative Importance Variable Method with $n_d = 3$ (one iteration)	3·590	$0{\cdot}165 \times 10^{-3}$
Conventional FORM with full set of six variables	3·590	$0{\cdot}165 \times 10^{-3}$
MCS with 30,000 samples	–	$0{\cdot}166 \times 10^{-3}$

surface assumed to be:

$$g(\mathbf{x}) = d_{\max} - |\delta v_3(\mathbf{x})|, \tag{37}$$

where the random variable δv_3 is the vertical displacement at node 3. The structure is unsafe when $g(\mathbf{x}) < 0$ that is, when $\delta v_3 > d_{\max}$. For numerical calculations it is assumed that $d_{\max} = 0{\cdot}095$.

Numerical results obtained by using the proposed methods are shown in table 4. The Monte Carlo simulation (MCS) is performed with 30000 samples. The methods in table 4 are arranged in the order of increasing computational cost. It is clear that the all the proposed methods produce satisfactory agreement with usual FORM and the Monte Carlo simulation (considered as benchmark).

7.2 *Multistoried portal frame*

We consider a multistoried portal frame with 20 members. The details of the structure is shown in figure 9 with element numbering and node numbering.

It is assumed that the axial stiffness (EA) and the bending stiffness (EI) of each member are Gaussian random variables so that there are in total 40 random variables, $\mathbf{x} \in \mathbb{R}^{40}$. Like the previous example it is also assumed that EA and EI of different members are uncorrelated. The joint probability density function (pdf) of the random variables is completely characterized by the mean and the standard deviation of EA and EI of each member. There are three types of elements and the numerical values of the properties of each element type is shown in table 5.

Table 5. Element types and associated elements numbers of the random multistoried portal frame as in figure 9. The standard deviations are expressed as a percentage of the corresponding mean values.

	EA (KN)		EI (KNm^2)		
Element type	Mean	Standard deviation	Mean	Standard deviation	Element numbers
1	$5{\cdot}0 \times 10^9$	7·0%	$6{\cdot}0 \times 10^4$	5·0%	1,3,5,7,9,11,13,15,17,19
2	$3{\cdot}0 \times 10^9$	3·0%	$4{\cdot}0 \times 10^4$	10·0%	2,6,10,14,18
3	$1{\cdot}0 \times 10^9$	10·0%	$2{\cdot}0 \times 10^4$	9·0%	4,8,12,16,20

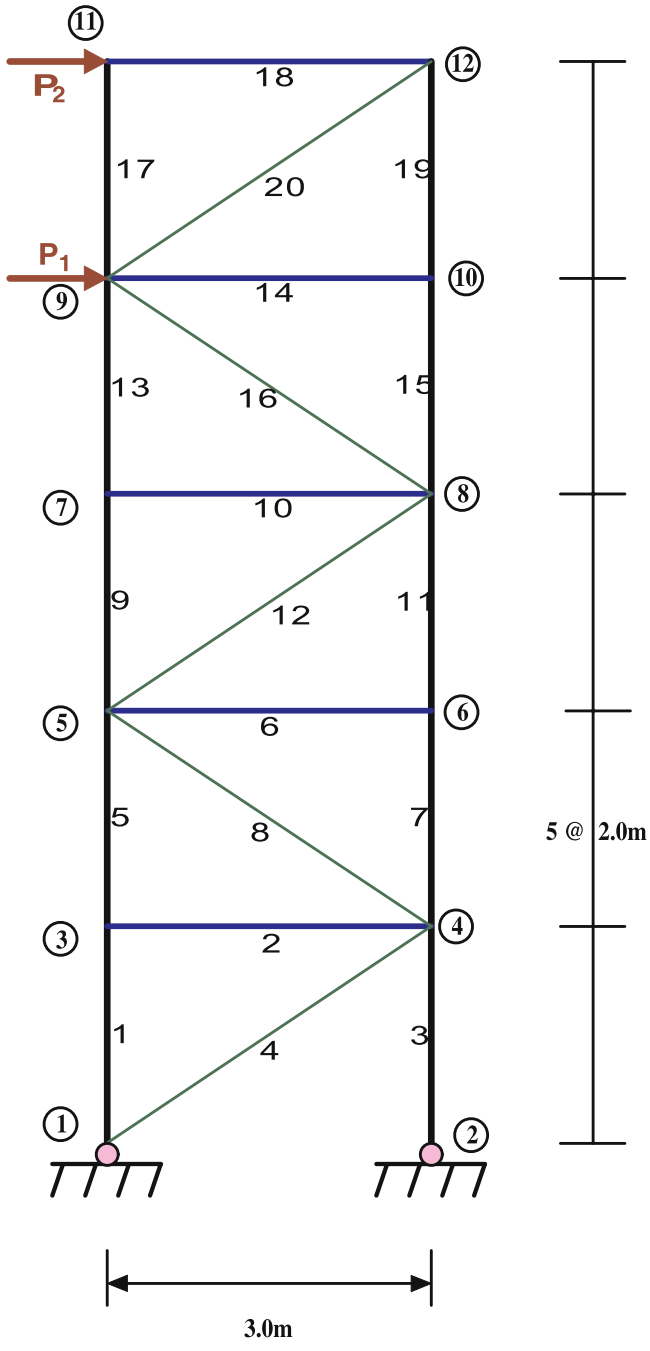

Figure 9. Multistoried portal with random element properties, $P_1 = 4{\cdot}0 \times 10^5$ KN and $P_2 = 5{\cdot}0 \times 10^5$ KN.

It is considered that the column members are of type 1, the beam members are of type 2 and the bracing members are of type 3.

Two horizontal forces $P_1 = 4{\cdot}0 \times 10^5$ KN and $P_2 = 5{\cdot}0 \times 10^5$ KN are applied at nodes 9 and 11 respectively. The failure condition is given by specifying a maximum allowable horizontal

Table 6. Comparison of reliability index and failure probability of the random multistoried portal frame as in figure 9.

Method	β	P_f
Gradient Iteration Method (one iteration)	3·399	$0{\cdot}338 \times 10^{-3}$
Dominant Gradient Method with $n_d = 5$ (one iteration)	3·397	$0{\cdot}340 \times 10^{-3}$
Relative Importance Variable Method with $n_d = 5$ (one iteration)	3·397	$0{\cdot}340 \times 10^{-3}$
Conventional FORM with full set of 40 variables	3·397	$0{\cdot}340 \times 10^{-3}$
MCS with 11600 samples	–	$0{\cdot}345 \times 10^{-3}$

displacement at node 11, say $d_{\max}$. We construct the failure surface

$$g(\mathbf{x}) = d_{\max} - |\delta h_{11}(\mathbf{x})|, \tag{38}$$

where the random variable δh_{11} is the horizontal displacement at node 11. The structure is unsafe when $g(\mathbf{x}) < 0$ that is, when $\delta h_{11}(\mathbf{x}) > d_{\max}$. For numerical calculations it is assumed that $d_{\max} = 1{\cdot}84 \times 10^{-3}$ m. Numerical results obtained using the proposed methods are shown in table 6. The Monte Carlo simulation (MCS) is performed with 11600 samples. The methods in table 6 are arranged in the order of increasing computational cost. It is clear that the all the proposed methods using reduced number of random variables produce same result obtained by conventional FORM with full set of 40 variables. Further, also note that all the approximate reliability estimate methods show satisfactory agreement with the Monte Carlo simulation (MCS) (considered as benchmark).

8. Conclusions

New methods have been proposed to reduce the number of random variables in structural reliability problems involving a large number of random variables. In total three iterative methods, namely (a) gradient iteration method, (b) dominant gradient method, and (c) relative importance variable method, have been proposed. All the three methods are simple, intuitive and based on the sensitivity vector of the failure surface. Once the design point and the reliability index are obtained using one of the proposed approaches, several methods are available to obtain the failure probability accurately. Initial numerical results show that there may be a possibility to put these methods into real-life problems involving a large number of random variables. Further studies will involve analysing the efficiency of the proposed methods when applied to problems with highly nonlinear failure surfaces, such as in dynamic problems.

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