On Symmetrizable Systems of Second Kind

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We discuss under what conditions multiple-parameter asymmetric linear dynamical systems can be transformed into equivalent symmetric systems by nonsingular linear transformations. So far, in structural dynamics literature this problem has been addressed in the context of the original work by Taussky. Taussky's approach of symmetrization was based on similarity transformation. In this paper an approach is proposed to transform asymmetric systems into symmetric systems by equivalence transformation. We call Taussky's approach of symmetrization by similarity transformation "first kind" and proposed approach by equivalence transformation "second kind." Since equivalence transformations are most general nonsingular linear transformations, conditions of symmetrizability obtained here are more "liberal" than the first kind and numerical calculations also become more straightforward. Several examples are provided to illustrate the new approach. [S0021-8936(00)00504-3]

1 Introduction

Theory of linear dynamics of multiple parameter symmetric systems is well developed now. However, dynamical behavior of some systems encountered in practice cannot be expressed in terms of symmetric coefficient matrices or self-adjoint linear operators. Some examples are gyroscopic and circulatory systems ([1]), aircraft flutter ([2]), ship motion in sea water ([3]), contact problems ([4]), and many actively controlled systems ([5]). Few authors have considered such general asymmetric dynamical systems. Fawzy and Bishop [6] presented several relationships satisfied by the eigenvectors and "eigenrows" of a damped asymmetric system and also presented a method to normalize them. Caughey and Ma [5] have derived conditions under which such systems can be diagonalized by a *similarity transformation.* In a subsequent paper Ma and Caughey [7] utilized *equivalence transformation* to analyze asymmetric nonconservative systems and gave the condition under which they can be diagonalized. Recently Adhikari [8] proposed a method to obtain (complex) eigensolutions of general asymmetric nonconservative systems without converting the equations of motion into first-order form.

The above-mentioned works have certainly enhanced the power of modal analysis in dealing with asymmetric systems. However, asymmetric systems are still not as well understood as symmetric systems. For example, rightly pointed out by Kliem [9], stability investigations become substantially easier if the system matrices are symmetric. Without doubt, it would be preferable if asymmetric systems can be transformed into equivalent symmetric systems so that one can take advantage of the well-developed theories for symmetric systems to analyze them. This is the primary reason to study "symmetrizability" of asymmetric systems. In linear dynamics literature symmetrizability has been addressed by *similarity transformation* based on Taussky's [10] definition. We call Taussky's approach of symmetrization by similarity transformation as "symmetrization of first kind." In this paper symmetrizability of a matrix is redefined in the context of *equivalence transformation.* Such symmetrization will be called "symmetrization of second kind."

Equivalence transformations are the most general class of nonsingular linear transformations. This motivates us to utilize equivalence transformation rather than similarity transformation for symmetrization of asymmetric systems. It will be shown that much generality can be achieved by using symmetrization of the second kind compared to the first kind. Notations and definitions of some terminologies frequently used in this paper are given in Section 2. Taussky's version of symmetrization and a brief review of the literature on its applications in structural dynamics is presented in Section 3. In Section 4 we formally define the symmetrization of the second kind. For the sake of generality, some basic results on such symmetrization are presented on complex matrices. Numerical methods are outlined to carry out such a symmetrization procedure. In view of the undamped and damped dynamical systems, simultaneous symmetrization of two and three matrices are considered in Section 5 and Section 6, respectively. Finally, Section 7 summarizes the main results of this paper and some suggestions towards further research required to successfully apply this new approach is provided. Throughout the paper suitable numerical examples are provided to illustrate the derived results.

2 Notations, Basic Concepts, and Definitions

By $\mathbb{R}^{N\times N}$ we mean the space of $N\times N$ real matrices and $\mathbb{C}^{N\times N}$ stands for the space of $N\times N$ complex matrices. An $N\times N$ matrix taken from either real or complex number field will be denoted by $\mathbb{F}^{N\times N}$. A matrix $\mathbf{A}$ is called positive definite if all of its eigenvalues $\lambda_i>0$ and will be denoted by $\mathbf{A}>0$. A unit matrix $\mathbf{I}\in\mathbb{R}^{N\times N}$ is a diagonal matrix with all diagonal entries equal to one. Let $\mathbf{A}\in\mathbb{C}^{N\times N}$, then we denote $\mathbf{A}^T$, $\overline{\mathbf{A}}$, $\mathbf{A}^{-1}$, $\mathbf{A}^{-T}$, and $\mathbf{A}^*$ be the transpose, complex conjugate, inverse, inverse transposed, and transposed conjugate. A matrix $\mathbf{A}$ is called *symmetric* if $\mathbf{A}=\mathbf{A}^T$, *Hermitian* if $\mathbf{A}=\mathbf{A}^*$, and *unitary* if $\mathbf{A}\mathbf{A}^*=\mathbf{I}$. If $\mathbf{A}$ is real then a Hermitian matrix is equivalent to a symmetric matrix and a unitary matrix is equivalent to a real orthogonal matrix as $\mathbf{A}\mathbf{A}^T=\mathbf{I}$. For a matrix $\mathbf{A}$, by saying $\mathbf{A}^{-1}$ exists we mean none of its eigenvalues is equal to 0 and that $\mathbf{A}$ is nonsingular. Two matrices, $\mathbf{A}$ and $\mathbf{B}$, related by $\mathbf{B}=\mathbf{V}^T\mathbf{A}\mathbf{U}$ for some nonzero $\mathbf{U}$ and $\mathbf{V}$, are called an *equivalence transformation.* When $\mathbf{V}^T=\mathbf{U}^{-1}$, the equivalence transformation is called the *similarity transformation* and we call $\mathbf{A}$ and $\mathbf{B}$ *similar.* In the event $\mathbf{V}=\mathbf{U}$ the equivalence transformation is a *congruence transformation.* Classical modal transformation in symmetric systems is an example of congruence transformation. When $\mathbf{V}^T=\mathbf{U}^{-1}=\mathbf{U}^T$, we call such transformation the *orthogonal transformation.* Further, if $\mathbf{V}^T=\mathbf{V}^{-1}$ and $\mathbf{U}^T=\mathbf{U}^{-1}$, such transformation is called the *biorthogonal transformation.* A

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, Sept. 21, 1999; final revision, May 2, 2000. Associate Technical Editor: A. A. Ferri. Discussion on the paper should be addressed to the Technical Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.

matrix $\mathbf{A}$ is said to be *diagonalizable* if it is similar to a diagonal matrix and that diagonal matrix contains all the eigenvalues of $\mathbf{A}$ with proper multiplicities. A matrix is diagonalizable if and only if it has N linearly independent eigenvectors.

3 Symmetrizability of the First Kind

The concept of symmetrizability of an asymmetric matrix was originated from an excellent result by Taussky and Zassenhaus [11] which says "for every $\mathbf{A}\in\mathbb{F}^{N\times N}$ there is a non-singular symmetric matrix transforming $\mathbf{A}$ into its transpose." Based on this general result it was shown that every real square matrix can be expressed as a product of two real symmetric matrices, that is

$$\mathbf{A}=\mathbf{S}_1\mathbf{S}_2;\quad \mathbf{S}_i=\mathbf{S}_i^T\in\mathbb{R}^{N\times N}, \tag{3.1}$$

always holds. Later Taussky [10] proved that if one of the factors in this representation is positive definite then $\mathbf{A}$ can be transformed into a symmetric matrix by a similarity transformation and vice versa. She formally defined *symmetrizability* of a matrix as the following:

DEFINITION 1. *A matrix* $\mathbf{A}$ *is symmetrizable if and only if any one of the following hold:*

1 $\mathbf{A}$ is the product of two symmetric matrices, one of which is positive definite;
2 $\mathbf{A}$ is similar to a symmetric matrix;
3 $\mathbf{A}^T=\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ with $\mathbf{S}=\mathbf{S}^T>0$; and
4 $\mathbf{A}$ has real characteristic roots and a full set of characteristic vectors.

Huseyin and Liepholz [1] were possibly first to recognize the importance of Taussky's result in the context of structural dynamics. However, symmetrization was not properly exploited until a decade later when Inman's [12] paper appeared. Inman's work inspired several authors to consider the dynamics of symmetrizable asymmetric systems. Subsequent works by Ahmadian and Inman [13,14], Ahmadian and Chou [15], Shahruz and Ma [16], and Cherng and Abdelhamid [17] made significant contributions and symmetrization of the first kind is much better understood now. Next, a more general approach for symmetrization is proposed.

4 Symmetrizability of the Second Kind

4.1 Definition. Now we will introduce a new definition of symmetrizability by utilizing equivalence transformation.

DEFINITION 2. *A matrix* $\mathbf{A}$ *is symmetrizable of the second kind if and only if there exist two nonzero matrices* $\mathbf{L}$ *and* $\mathbf{R}$ *such that* $\tilde{\mathbf{A}}=\mathbf{L}^T\mathbf{A}\mathbf{R}$ *is a symmetric matrix.*

This definition of symmetrizability is quite general and valid for both real and complex matrices. It also holds Taussky's definition as a special case when $\mathbf{L}^T=\mathbf{R}^{-1}=\mathbf{R}^{-T}$. In the above definition, even $\mathbf{A}$ is real $\mathbf{L}$ and $\mathbf{R}$ might be complex in order to make $\tilde{\mathbf{A}}$ symmetric. For any $\mathbf{A}$, when $\mathbf{L}$ and $\mathbf{R}$ are real matrices we call such $\mathbf{A}$ *real symmetrizable* and *complex symmetrizable* when $\mathbf{L}$ and $\mathbf{R}$ are complex. Taussky [18] and Pommer and Kliem [19] have discussed complex symmetrizable matrices in the context of a similarity transformation. Applications of such complex symmetrizability in linear dynamical systems were given by Kliem [9].

4.2 Basic Results. Some basic results on symmetrizability of the second kind and a numerical method to carry out such a symmetrization procedure will be developed. In order to achieve more generality, one of our main results is presented for nonsquare complex matrices in the following theorem:

THEOREM 4.1. *Every rectangular complex matrix* $\mathbf{A}\in\mathbb{C}^{r\times s}$ *can be transformed to a real symmetric (square) matrix by an equivalence transformation.*

Proof. We use a result by Eckrat and Young ([20], Theorem 1) which states that *for every* $\mathbf{A}\in\mathbb{C}^{r\times s}$ *there are two unitary matrices* $\mathbf{X}\in\mathbb{C}^{r\times r}$ *and* $\mathbf{Y}\in\mathbb{C}^{s\times s}$ *such that* $\mathbf{D}=\mathbf{X}^*\mathbf{A}\mathbf{Y}$ *is a diagonal matrix with real elements, none of which are negative.* If $\text{rank}(\mathbf{A})=k$, then Williamson [21] has shown that $\mathbf{D}=\left[\begin{smallmatrix}\mathbf{\Lambda} & \mathbf{0}_1\\ \mathbf{0}_2 & \mathbf{0}_3\end{smallmatrix}\right]$ where $\mathbf{\Lambda}=\text{diag}(\lambda_1,\cdots,\lambda_k)$ is a real diagonal matrix and $\mathbf{0}_1$, $\mathbf{0}_2$, and $\mathbf{0}_3$ are $(k,s-k)$, $(r-k,k)$, and $(r-k,s-k)$ null matrices. Without loss of generality we select $\mathbf{L}=\overline{\mathbf{X}}\mathbf{Q}_1$ and $\mathbf{R}=\mathbf{Y}\mathbf{Q}_2$ for some nonzero $\mathbf{Q}_1\in\mathbb{R}^{r\times n}$ and $\mathbf{Q}_2\in\mathbb{R}^{s\times n}$ such that $\mathbf{Q}_1=\left[\begin{smallmatrix}\mathbf{\Theta} & \mathbf{Q}_1^1\\ \mathbf{Q}_2^1 & \mathbf{Q}_3^1\end{smallmatrix}\right]$ and $\mathbf{Q}_2=\left[\begin{smallmatrix}\mathbf{\Theta} & \mathbf{Q}_1^2\\ \mathbf{Q}_2^2 & \mathbf{Q}_3^2\end{smallmatrix}\right]$ where $\mathbf{\Theta}\in\mathbb{R}^{k\times k}$ and $\mathbf{Q}_j^i$, $i=1,2$; $j=1,2,3$ are all real matrices of proper orders. Using these, one has $\tilde{\mathbf{A}}=\mathbf{L}^T\mathbf{A}\mathbf{R}=\mathbf{Q}_1^T\mathbf{X}^*\mathbf{A}\mathbf{Y}\mathbf{Q}_2=\mathbf{Q}_1^T\mathbf{D}\mathbf{Q}_2=\left[\begin{smallmatrix}\mathbf{\Theta}^T\mathbf{\Lambda}\mathbf{\Theta} & \tilde{\mathbf{0}}_1\\ \tilde{\mathbf{0}}_2 & \mathbf{0}_3\end{smallmatrix}\right]\in\mathbb{R}^{n\times n}$ where $\tilde{\mathbf{0}}_j$; $j=1,2,3$ are all null matrices with proper orders. So $\tilde{\mathbf{A}}$ is real symmetric matrix and this completes the proof. □

Remark 4.1. The two matrices $\mathbf{Q}_1$ and $\mathbf{Q}_2$ utilized above can be selected in such a way that the symmetric form becomes nonsingular and has the same rank as $\mathbf{A}$. Observe that because $\mathbf{D}$ is a diagonal matrix with non-negative real elements, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can also be selected in a way such that the symmetric form becomes positive definitive. The symmetric form of $\mathbf{A}$ is, however, not unique because $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be chosen in several ways.

This result clearly demonstrates the generality of the symmetrization of the second kind compared to the first kind which cannot be applied to rectangular matrices. Of course, Theorem 4.1 is applicable for real matrices as a special case. When $\mathbf{A}$ is a real matrix we have the following interesting special result:

THEOREM 4.2. *Every real matrix* $\mathbf{A}$ *is real symmetrizable of the second kind.*

Proof. We use a result by Pearl ([22], Theorem 1). This gives the condition under which the unitary matrices which reduces the matrix $\mathbf{A}$ to a real diagonal form by an equivalence transformation in Eckrat and Young's theorem are real orthogonal matrices. According to this result, *for* $\mathbf{A}\in\mathbb{C}^{r\times s}$ *there are two real orthogonal matrices* $\mathbf{O}_1\in\mathbb{R}^{r\times r}$ *and* $\mathbf{O}_2\in\mathbb{R}^{s\times s}$ *such that* $\mathbf{D}=\mathbf{O}_1^*\mathbf{A}\mathbf{O}_2$ *is a diagonal matrix with real non-negative elements if and only if* $\mathbf{A}\mathbf{A}^*$ *and* $\mathbf{A}^*\mathbf{A}$ *are both real matrices.* The theorem is proved since (trivially) this condition is always satisfied when $\mathbf{A}$ is real. □

We have described two important results on symmetrization of the second kind. From a practical point of view the obvious question is, for a given matrix $\mathbf{A}$, how to calculate $\mathbf{L}$ and $\mathbf{R}$ so that $\tilde{\mathbf{A}}=\mathbf{L}^T\mathbf{A}\mathbf{R}$ is a real symmetric matrix? In answer to this question, we observe that the proof given by Eckrat and Young [20] of their theorem described previously can be used for obtaining $\mathbf{L}$ and $\mathbf{R}$. For computational purposes the following steps may be followed to transform a general complex matrix $\mathbf{A}\in\mathbb{C}^{r\times s}$ to a nonsingular real square matrix using an equivalence transformation.

1 Calculate the matrix $\mathcal{M}=\mathbf{A}\mathbf{A}^*\in\mathbb{C}^{r\times r}$. Solve the eigenvalue problem $\mathcal{M}\mathbf{x}_i=\lambda_i^2\mathbf{x}_i$, $\forall i=1,\cdots r$. Since $\mathcal{M}$ is a non-negative Hermitian its eigenvector matrix can be normalized to a unitary matrix. Suppose $\text{rank}(\mathbf{A})=k$, arrange the numbering of the eigenvalues so that $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_k>0$; $\lambda_{k+1}=\cdots=\lambda_r=0$. Denote $\mathbf{X}=[\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_k]\in\mathbb{C}^{r\times k}$ as the ordered collection of the eigenvectors.

2 Evaluate $\mathbf{y}_i=\mathbf{A}^*\mathbf{x}_i/\lambda_i$, $\forall i=1,\cdots k$. Set $\mathbf{Y}=[\mathbf{y}_1\mathbf{y}_2,\cdots,\mathbf{y}_k]\in\mathbb{C}^{s\times k}$.

3 Consider any nonzero $\mathbf{Q}\in\mathbb{R}^{k\times k}$ and obtain $\mathbf{L}=\overline{\mathbf{X}}\mathbf{Q}\in\mathbb{C}^{r\times k}$ and $\mathbf{R}=\mathbf{Y}\mathbf{Q}\in\mathbb{C}^{s\times k}$. Finally check that $\tilde{\mathbf{A}}=\mathbf{L}^T\mathbf{A}\mathbf{R}$ is a $(k\times k)$ real nonsingular symmetric matrix.

For the sake of generality the method is proposed for rectangular complex matrices. This procedure is obviously applicable to real square matrices we normally encounter in the equations of motion of linear systems. When $\mathbf{A}$ is real, $\mathcal{M}$ becomes a real symmetric matrix and consequently $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{L}$, and $\mathbf{R}$ all become real matrices. For a further special case, when $\mathbf{A}$ is a symmetrizable matrix of first kind, this procedure provides an alternative and easy way to find symmetric forms as the calculation of Taussky's factorization can be avoided.

Example 4.1. The procedure outlined above can be illustrated by the following rectangular complex matrix:

$$\mathbf{A}=\begin{bmatrix} 1.0+3.0i & -2.0+1.0i & 1.5-2.0i \\ -2.0-3.0i & 6.0+2.0i & 7.0+2.0i \end{bmatrix}. \tag{4.1}$$

Step 1: We calculate $\mathcal{M}=\mathbf{AA}^*=\left[\begin{smallmatrix} 21.25 & -14.50-10.00i \\ -14.50+10.00i & 106.00 \end{smallmatrix}\right]$. Solving the eigenvalue problem $\mathcal{M}\mathbf{x}_i=\lambda_i^2\mathbf{x}_i$ and arranging accordingly one has $\lambda_i^2=\{109.5150,\ 17.7350\}$ and $\mathbf{X}=\left[\begin{smallmatrix} -0.1611-0.1111i & 0.9807 \\ 0.9807 & 0.1611-0.1111i \end{smallmatrix}\right]$. Note that $\text{rank}(\mathbf{A})=2$.

Step 2: Calculating one obtains

$$\mathbf{Y}=\begin{bmatrix} -0.2347+0.3167i & 0.2355-0.5311i \\ 0.5824-0.1508i & -0.2890-0.4677i \\ 0.6541-0.2341i & 0.5643+0.2045i \end{bmatrix}.$$

Step 3: Now arbitrarily select the matrix $\mathbf{Q}=\left[\begin{smallmatrix} 0.1021 & 0.2605 \\ 1.4109 & 0.3519 \end{smallmatrix}\right]$ so that $\mathbf{L}=\overline{\mathbf{X}}\mathbf{Q}=\left[\begin{smallmatrix} 1.3672+0.0113i & 0.3031+0.0289i \\ 0.3274+0.1568i & 0.3121+0.0391i \end{smallmatrix}\right]$ and

$$\mathbf{R}=\mathbf{YQ}=\begin{bmatrix} 0.3083-0.7170i & 0.0218-0.1044i \\ -0.3483-0.6752i & 0.0500-0.2039i \\ 0.8630+0.2647i & 0.3690+0.0110i \end{bmatrix}.$$

Using this $\mathbf{L}$ and $\mathbf{R}$ one has $\tilde{\mathbf{A}}=\mathbf{L}^T\mathbf{AR}=\left[\begin{smallmatrix} 8.4923 & 2.3692 \\ 2.3692 & 1.2315 \end{smallmatrix}\right]$, which is a real nonsingular symmetric matrix.

Example 4.2. Consider a real asymmetric matrix

$$\mathbf{A}=\begin{bmatrix} 1.0 & -2.0 & 1.5 \\ 12.0 & 6.0 & 7.0 \\ -2.0 & 4.0 & 9.0 \end{bmatrix}. \tag{4.2}$$

Observe that $\text{rank}(\mathbf{A})=3$. Selecting the matrix

$$\mathbf{Q}=\begin{bmatrix} 0.0645 & -0.1647 & 0.1436 \\ -0.8923 & 0.2226 & 0.6155 \\ 0.1611 & 0.2298 & 0.3708 \end{bmatrix} \tag{4.3}$$

and following the procedure described previously we obtain

$$\mathbf{L}=\begin{bmatrix} 0.1735 & 0.2198 & 0.3701 \\ 0.3901 & -0.2469 & -0.1170 \\ -0.8026 & 0.1421 & 0.6215 \end{bmatrix}$$

and (4.4)

$$\mathbf{R}=\begin{bmatrix} 0.7299 & -0.2298 & -0.2942 \\ -0.2638 & -0.2377 & -0.1620 \\ -0.4733 & 0.1419 & 0.6513 \end{bmatrix}$$

and both are real matrices. From these one has

$$\tilde{\mathbf{A}}=\mathbf{L}^T\mathbf{AR}=\begin{bmatrix} 7.0392 & -1.7960 & -4.4600 \\ -1.7960 & 1.0004 & 1.0316 \\ -4.4600 & 1.0316 & 3.9722 \end{bmatrix}$$

a real nonsingular symmetric matrix. Interestingly, note that $\text{eig}(\mathbf{A})=\{12.8914,\ 1.5543+5.0507i,\ 1.5543-5.0507i\}$, i.e., $\mathbf{A}$ does not satisfy Taussky's condition of symmetrizability (Definition 1). Thus $\mathbf{A}$ in (4.2) is a real symmetrizable matrix of the second kind but not a real symmetrizable of the first kind. This illustrates generality of the proposed approach of symmetrization compared to the conventional approach.

Because the coefficient matrices in the equations of motion of linear vibrations are real we next consider only real square matrices.

5 Simultaneous Symmetrization of Two Matrices

Equations of motion of a linear undamped (nongyroscopic) system can be expressed by

$$\mathbf{A}\ddot{\mathbf{x}}(t)+\mathbf{B}\mathbf{x}(t)=\mathbf{0}, \quad t>0, \tag{5.1}$$

where $\mathbf{A}\in\mathbb{R}^{N\times N}$ and $\mathbf{B}\in\mathbb{R}^{N\times N}$ and $\mathbf{x}(t)\in\mathbb{R}^N$ is the vector of the *generalized coordinates.* Were $\mathbf{A}$ and $\mathbf{B}$ symmetric and positive definite matrices they would, respectively, represent the mass and stiffness matrices. Here, however, no such restrictions are imposed on $\mathbf{A}$ and $\mathbf{B}$. Huseyin [23] has shown that the traditional modal analysis, originally developed for symmetric systems, can be extended to solve Eq. (5.1). There exist *right eigenvectors* or *normal modes* and *left eigenvectors* or *adjoint modes* which uncouple the equations of motion into N single-degree-of-freedom oscillators. Although system (5.1) can always be solved using normal modes and adjoint modes, it would be more useful if it can be transformed to a symmetric system. We consider the following two cases to investigate symmetrizability of system (5.1).

5.1 Case 1: A and B Have Real Roots and A is Nonsingular.

This is the most common case we encounter in practice. If system (5.1) is ''truly'' undamped then $\mathbf{A}$ or $\mathbf{B}$ does not possess any complex roots because, otherwise a stable system will not have periodic motion for an infinitely long time but vibration (in some degrees-of-freedom) will decay due to the complex nature of the eigenvalues. As $\mathbf{A}$ is nonsingular, rewriting Eq. (5.1) one has

$$\mathbf{I}\ddot{\mathbf{x}}+\mathbf{E}\mathbf{x}=\mathbf{0} \tag{5.2}$$

where $\mathbf{E}=\mathbf{A}^{-1}\mathbf{B}\in\mathbb{R}^{N\times N}$ has real roots as both $\mathbf{A}$ and $\mathbf{B}$ have real roots. Because this matrix satisfies Condition 4 of Taussky's definition of symmetrizability, system (5.2) can be transformed to a symmetric system by a similarity transformation. From this discussion we have the following interesting result:

THEOREM 5.1. *All asymmetric undamped systems are similar to symmetric undamped systems.*

Practical implementation of Taussky's factorization (3.1) is not straightforward (see Ahmadian and Chou, [15], Shahruz and Ma [16], and Cherng and Abdelhamid [17]). For this reason it is required to develop efficient numerical methods for finding associated symmetric form(s) of system (5.1). Recall that as system (5.1) is symmetrizable of the first kind it is also symmetrizable of the second kind. We use the latter approach to avoid the calculation of Taussky's factorization.

It has been mentioned that there always exist two matrices $\mathbf{U}\in\mathbb{R}^{N\times N}$ (normal modal matrix) and $\mathbf{V}\in\mathbb{R}^{N\times N}$ (adjoint modal matrix) such that $\mathbf{V}^T\mathbf{AU}$ and $\mathbf{V}^T\mathbf{BU}$ are both real diagonal matrices. Numerical methods for obtaining $\mathbf{U}$ and $\mathbf{V}$ are well developed (see Huseyin [23] and Ma and Caughey [7] for further discussions). Select $\mathbf{L}=\mathbf{VQ}$ and $\mathbf{R}=\mathbf{UQ}$ for some nonzero $\mathbf{Q}\in\mathbb{R}^{N\times N}$. Clearly $\mathbf{L}^T\mathbf{AR}$ and $\mathbf{L}^T\mathbf{BR}$ are both symmetric matrices now. This approach is much easier for finding symmetric forms than the approach via Taussky's factorization. Also note that symmetric forms are nonunique since one can select the matrix $\mathbf{Q}$ in many ways. The following example demonstrates this procedure.

Example 5.1. Suppose the coefficient matrices of an undamped system of the form (5.1) are given by

$$\mathbf{A}=\begin{bmatrix} 0.5740 & 1.3858 & 1.3858 \\ 0.7070 & 0.7070 & -0.7070 \\ 0.4620 & -0.1914 & -0.1914 \end{bmatrix}$$

and

$$\mathbf{B}=\begin{bmatrix} 1.3748 & 10.9440 & 25.2975 \\ 1.2625 & 2.8770 & -17.4195 \\ 0.7455 & -4.1244 & 0.8625 \end{bmatrix}. \tag{5.3}$$

Numerical values of $\mathbf{A}$ and $\mathbf{B}$ are taken from Ma and Caughey [7]. It may be verified that $\mathbf{A}$, $\mathbf{B}$ and consequently $\mathbf{E}=\mathbf{A}^{-1}\mathbf{B}$ are symmetrizable matrices of first kind.

Solving the right and left eigenvalue problems $\mathbf{BU}=\mathbf{\Lambda AU}$ and $\mathbf{V}^T\mathbf{B}=\mathbf{\Lambda V}^T\mathbf{A}$ we obtain the undamped modal matrices

$$\mathbf{U}=\begin{bmatrix} 0.9996 & -0.5434 & 0.3951 \\ -0.0271 & 0.8391 & -0.4850 \\ -0.0044 & -0.0253 & 0.7802 \end{bmatrix}$$

and

$$\mathbf{V}=\begin{bmatrix} 0.2868 & 0.4247 & 0.4617 \\ 0.4677 & 0.2613 & -0.7124 \\ 0.8361 & -0.8668 & 0.5285 \end{bmatrix} \quad (5.4)$$

and the natural frequencies squared $\mathrm{diag}(\mathbf{\Lambda})=\{1.8241, 9.5561, 23.5486\}$. Selecting the matrix $\mathbf{Q}$ given by (4.3) one obtains

$$\mathbf{L}=\mathbf{VQ}=\begin{bmatrix} -0.2860 & 0.1534 & 0.4737 \\ -0.3177 & -0.1826 & -0.0362 \\ 0.9126 & -0.2092 & -0.2176 \end{bmatrix}$$

and

$$\mathbf{R}=\mathbf{UQ}=\begin{bmatrix} 0.6131 & -0.1948 & -0.0445 \\ -0.8286 & 0.0798 & 0.3328 \\ 0.1480 & 0.1744 & 0.2731 \end{bmatrix}. \quad (5.5)$$

Using these matrices we have

$$\tilde{\mathbf{A}}=\mathbf{L}^T\mathbf{AR}=\begin{bmatrix} 0.6282 & -0.1303 & -0.3608 \\ -0.1303 & 0.1032 & 0.1514 \\ -0.3608 & 0.1514 & 0.4150 \end{bmatrix}$$

and

$$\tilde{\mathbf{B}}=\mathbf{L}^T\mathbf{BR}=\begin{bmatrix} 6.2653 & -0.7600 & -2.8429 \\ -0.7600 & 1.3947 & 2.5574 \\ -2.8429 & 2.5574 & 5.3631 \end{bmatrix}. \quad (5.6)$$

Above is a symmetric form of the asymmetric system (5.3). Note that symmetric form of this system can also be obtained alternatively by calculating Taussky's factorization of $\mathbf{E}$.

5.2 Case 2: A and B Are General Matrices. We have shown that when $\mathbf{A}$ and $\mathbf{B}$ have real roots, system (5.1) has an equivalent symmetric form via real linear transformations. However, if these matrices have complex roots, the matrix $\mathbf{E}$ in Eq. (5.2) in general does not satisfy Taussky's definition of symmetrizability. In that case, can system (5.1) be transformed to a real symmetric system using a real linear transformation? Our answer is the following:

LEMMA 5.2. *Linear undamped system* (5.1) *is real symmetrizable if* $\mathbf{A}^T\mathbf{B}$ *and* $\mathbf{BA}^T$ *are symmetric matrices.*

Proof. According to Thompson [24] "there exist two real orthogonal matrices $\mathbf{O}_1\in\mathbb{R}^{N\times N}$ and $\mathbf{O}_2\in\mathbb{R}^{N\times N}$ such that square matrices $\mathbf{O}_1^T\mathbf{AO}_2=\mathbf{\Lambda}_a$ and $\mathbf{O}_1^T\mathbf{BO}_2=\mathbf{\Lambda}_b$ are diagonal matrices if and only if $\mathbf{A}^T\mathbf{B}$ and $\mathbf{BA}^T$ are symmetric matrices." Now construct $\mathbf{L}=\mathbf{O}_1\mathbf{Q}$ and $\mathbf{R}=\mathbf{O}_2\mathbf{Q}$ for some nonzero $\mathbf{Q}\in\mathbb{R}^{N\times N}$. Using these we have $\mathbf{L}^T\mathbf{AR}=\mathbf{Q}^T\mathbf{\Lambda}_a\mathbf{Q}$ and $\mathbf{L}^T\mathbf{BR}=\mathbf{Q}^T\mathbf{\Lambda}_b\mathbf{Q}$ are both symmetric matrices. This completes the proof. □

This result provides only a *sufficient* condition for simultaneous real symmetrizability of the general matrices $\mathbf{A}$ and $\mathbf{B}$. Lack of it does not necessarily preclude existence of a real symmetric form of system (5.1). The following examples illustrates this fact.

Example 5.2. Consider an undamped system with $\mathbf{A}$ same as Example 4.2 and

$$\mathbf{B}=\begin{bmatrix} 1.2044 & -5.4425 & 2.7013 \\ 2.1007 & 0.8393 & 0.1894 \\ -1.8393 & 0.8953 & 2.5087 \end{bmatrix}. \quad (5.7)$$

One may verify that $\mathbf{A}$ and $\mathbf{B}$ are not symmetrizable of the first kind. Observe that the matrices

$$\mathbf{A}^T\mathbf{B}=\begin{bmatrix} 30.0910 & 2.8380 & -0.0427 \\ 2.8380 & 19.5018 & 5.7688 \\ -0.0427 & 5.7688 & 27.9563 \end{bmatrix}$$

and

$$\mathbf{BA}^T=\begin{bmatrix} 16.1414 & 0.7063 & 0.1331 \\ 0.7063 & 31.5697 & 0.8607 \\ 0.1331 & 0.8607 & 29.8380 \end{bmatrix}$$

are both symmetric. Since the system matrices satisfy the condition of Lemma 5.2 the transforming matrices $\mathbf{L}$ and $\mathbf{R}$ (computed before in Eq. (4.4)) which symmetries $\mathbf{A}$ also symmetries $\mathbf{B}$ as

$$\mathbf{L}^T\mathbf{BR}=\begin{bmatrix} 2.8766 & -0.4670 & -1.4786 \\ -0.4670 & 0.5501 & 0.9468 \\ -1.4786 & 0.9468 & 2.1818 \end{bmatrix}$$

is a symmetric matrix.

Example 5.3. Consider another undamped system with $\mathbf{A}$ in the above example and

$$\mathbf{B}=\begin{bmatrix} 0.1955 & 1.8857 & -3.2199 \\ -2.1312 & -0.2236 & 0.3609 \\ 1.0378 & 1.9501 & -1.5606 \end{bmatrix}. \quad (5.8)$$

Observe that the matrices

$$\mathbf{A}^T\mathbf{B}=\begin{bmatrix} -27.4545 & -4.6975 & 4.2325 \\ -9.0269 & 2.6874 & 2.3629 \\ -5.2845 & 18.8142 & -16.3493 \end{bmatrix}$$

and

$$\mathbf{BA}^T=\begin{bmatrix} -8.4058 & -8.8789 & -21.8278 \\ -1.1426 & -24.3894 & 6.6164 \\ -5.2033 & 13.2300 & -8.3212 \end{bmatrix}$$

are not symmetric. However, the transforming matrices $\mathbf{L}$ and $\mathbf{R}$ obtained in Eq. (4.2) and also the symmetries $\mathbf{B}$ as

$$\mathbf{L}^T\mathbf{BR}=\begin{bmatrix} -1.2355 & 0.8082 & 1.2380 \\ 0.8082 & -0.4868 & -0.9953 \\ 1.2380 & -0.9953 & -2.0335 \end{bmatrix}$$

is a symmetric matrix. So the condition outlined in Lemma 5.2 is only sufficient.

6 Simultaneous Symmetrization of Three Matrices

The equations of motion describing free vibration of a viscously damped linear system can be expressed by

$$\mathbf{A}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{Bx}(t)=\mathbf{0}, \quad t>0. \quad (6.1)$$

We assume $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are $N\times N$ arrays of real numbers but are otherwise general. Were these matrices symmetric and positive definite, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ would, respectively, be the mass, stiffness, and viscous damping matrices. The equations of motion are now characterized by *three* real matrices and this brings an additional complication in the system dynamics. It is required to find a nonsingular linear transformation which simultaneously symmetries $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Unlike the undamped case, where in the absence of the **C** matrix conditions for obtaining such a transformation were easy to meet (Theorem 5.1), admissible forms of these matrices gets restricted here. We will investigate what forms of restrictions should be imposed on the system matrices so that the equations of motion can be transformed into a symmetric form. The following two cases are considered here.

6.1 Case 1: A is Nonsingular and All the Matrices Have Real Roots. When **A** is nonsingular, Eq. (6.1) can be rewritten as

$$\mathbf{I}\ddot{\mathbf{x}}(t)+\mathbf{F}\dot{\mathbf{x}}(t)+\mathbf{E}\mathbf{x}(t)=\mathbf{0}. \tag{6.2}$$

where $\mathbf{E}=\mathbf{A}^{-1}\mathbf{B}\in\mathbb{R}^{N\times N}$ and $\mathbf{F}=\mathbf{A}^{-1}\mathbf{C}\in\mathbb{R}^{N\times N}$. Since **A**, **B**, and **C** have real roots **E** and **F** also have real roots. Thus **E** and **F** are individually symmetrizable of the first kind. But this does not imply that system (6.2) is also symmetrizable, unless **I**, **E**, and **F** are simultaneously symmetrizable. A basic result in this regard is the following:

THEOREM 6.1. *All diagonalizable damped asymmetric systems are similar to symmetric systems.*

It is well known that system (6.2) is diagonalizable by a similarity transformation if and only if **E** and **F** commutes in multiplication, i.e., $\mathbf{EF}=\mathbf{FE}$. When this condition is satisfied the system is symmetrizable of the first kind and hence also symmetrizable of the second kind. Following the approach outlined before in Section 5.1, symmetric form(s) may be obtained by utilizing equivalence transformation to avoid the calculation of Taussky's factorization. A result similar to Theorem 6.1 was also established by Inman ([12], Theorem 1) and Shahruz and Ma ([16], Theorem 3.1). Note that the opposite statement of the above theorem, i.e., ''a damped asymmetric system similar to a symmetric system is diagonalizable'' is in general not true. Thus, diagonalizability of a system is only a sufficient condition for symmetrizability and rather restrictive. The following is a more stronger and liberal result:

THEOREM 6.2. (*Inman* [12], *Theorem* 2) *system* (6.2) *is symmetrizable if and only if* **E** *and* **F** *have a common symmetric positive definite factor.*

From Taussky's factorization (3.1) one can write $\mathbf{E}=\mathbf{S}_1\mathbf{S}_2$, $\mathbf{S}_1=\mathbf{S}_1^T>0$, $\mathbf{S}_2=\mathbf{S}_2^T$ and $\mathbf{F}=\mathbf{T}_1\mathbf{T}_2$, $\mathbf{T}_1=\mathbf{T}_1^T>0$, $\mathbf{T}_2=\mathbf{T}_2^T$. According to the above theorem system (6.2) is symmetrizable if and only if $\mathbf{S}_1=\mathbf{T}_1$. If this condition is satisfied Eq. (6.2) can be brought into symmetric form using a similarity transformation involving a *symmetric* matrix. For practical purposes this condition is difficult to check. A more convenient result is the following:

THEOREM 6.3. (*Kliem* [9], *Theorem 1*) *system* (6.2) *is real symmetrizable if and only if there exist modal matrices* $\mathbf{U}_E\in\mathbb{R}^{N\times N}$ *and* $\mathbf{U}_F\in\mathbb{R}^{N\times N}$ (*consisting of full eigenvector sets of* **E** *and* **F**) *such that* $\mathbf{U}_F^{-1}\mathbf{U}_E$ *is orthogonal, or equivalently* $\mathbf{U}_F\mathbf{U}_F^T=\mathbf{U}_E\mathbf{U}_E^T$.

Further discussions on this theorem can found in Pommer and Kliem [19]. When the condition of the above theorem is satisfied, Eq. (6.2) can be brought into symmetric form by a similarity transformation. In this case the symmetrizing matrix may be any nonzero matrix, unlike a symmetric matrix utilized in Theorem 6.2. For this reason Kliem's [9] condition is more liberal and holds Inman's [12] condition as a special case when the symmetrizing similarity transformation itself is a symmetric matrix. Thus Theorem 6.2, and consequently results obtained by Ahmadian and Chou ([15], Theorem 2) and Shahruz and Ma ([16], Theorem 4.1) based on this theorem are only *sufficient* conditions for symmetrization of the first kind of system (6.1).

Now, symmetrizability of system (6.1) by means of the equivalence transformation will be introduced. Our main result is the following:

THEOREM 6.4. *System* (6.1) *is real symmetrizable of the second kind if and only if there exist* $\mathbf{U}\in\mathbb{R}^{N\times N}$ *and* $\mathbf{V}\in\mathbb{R}^{N\times N}$, *the matrices of undamped right and left eigenvectors, such that* $\mathbf{V}^T\mathbf{C}\mathbf{U}$ *is symmetric.*

Proof. Because $\mathbf{U}\in\mathbb{R}^{N\times N}$ and $\mathbf{V}\in\mathbb{R}^{N\times N}$ are the matrices of right and left eigenvectors of the generalized eigenvalue problem involving **A** and **B**, $\mathbf{V}^T\mathbf{A}\mathbf{U}=\mathbf{\Lambda}_a$ and $\mathbf{V}^T\mathbf{B}\mathbf{U}=\mathbf{\Lambda}_b$ are both real diagonal matrices. Select $\mathbf{L}=\mathbf{VQ}$ and $\mathbf{R}=\mathbf{UQ}$, where $\mathbf{Q}\in\mathbb{R}^{N\times N}$ is nonzero. Clearly $\mathbf{L}^T\mathbf{A}\mathbf{R}=\mathbf{Q}^T\mathbf{\Lambda}_a\mathbf{Q}$ and $\mathbf{L}^T\mathbf{B}\mathbf{R}=\mathbf{Q}^T\mathbf{\Lambda}_b\mathbf{Q}$ are both real symmetric matrices. Using this transformation, the second term of Eq. (6.1) is now $\mathbf{L}^T\mathbf{C}\mathbf{R}=\mathbf{Q}^T(\mathbf{V}^T\mathbf{C}\mathbf{U})\mathbf{Q}$. Thus the transformed system is symmetric if and only if $\mathbf{V}^T\mathbf{C}\mathbf{U}$ is a symmetric matrix. □

Since the equivalence transformations are more general than similarity transformations it is expected that the condition of symmetrizability given by this theorem is less restrictive than the three previously established results discussed before. The following example illustrates this fact:

Example 6.1. Suppose, for a damped dynamic system, **A** and **B** are as in Example 5.1 and

$$\mathbf{C}=\begin{bmatrix}4.1429 & 9.4510 & 9.0014\\ 4.6465 & 3.8206 & -3.2816\\ 1.2006 & 0.5460 & -1.5001\end{bmatrix}. \tag{6.3}$$

Transforming the system in the form of Eq. (6.2) one may verify that both **E** and **F** are individually symmetrizable of the first kind. However, none of the conditions of simultaneous symmetrizability outlined in Theorems 6.1–6.3 is satisfied by **E** and **F.** Now, using the modal matrices **U** and **V** calculated before in Eq. (5.4) we have

$$\mathbf{V}^T\mathbf{C}\mathbf{U}=\begin{bmatrix}4.2300 & 1.7900 & -0.8400\\ 1.7900 & 26500 & 1.8900\\ -0.8400 & 1.8900 & 3.2100\end{bmatrix}$$

is a symmetric matrix, i.e., the condition of Theorem 6.4 is satisfied. The transformation matrices **L** and **R** given by Eq. (5.5) which symmetries **A** and **B** also symmetries **C** as

$$\mathbf{L}^T\mathbf{C}\mathbf{R}=\begin{bmatrix}1.4439 & -0.4735 & -1.8602\\ -0.4735 & 0.5412 & 0.8591\\ -1.8602 & 0.8591 & 2.6222\end{bmatrix}$$

is a symmetric matrix. Thus, the system under consideration is real symmetrizable of the second kind but not the first kind.

6.2 Case 2: A, B, and C are General Real Matrices. When the system matrices are general, some or all of them may have complex roots so that they do not satisfy Tausskkey's condition of symmetrizability. For such systems, in general Kliem's result (Theorem 6.3) gives the condition of *complex symmetrizability* of the first kind and Theorem 6.4 gives the condition of complex symmetrizability of the second kind. A *sufficient* condition for real symmetrizability of such systems is the following:

LEMMA 6.5. *System* (6.1) *is real symmetrizable of second kind if there exist* **U** *and* **V**, *the matrices of undamped right and left eigenvectors, such that* $\mathbf{A}^T\mathbf{B}$, $\mathbf{B}\mathbf{A}^T$ and $\mathbf{V}^T\mathbf{C}\mathbf{U}$ *are all symmetric matrices.*

This lemma can be proved easily following the results of Lemma 5.2 and Theorem 6.4. The example considered below illustrates this result:

Example 6.2. Suppose, for a damped linear system, **A** and **B** are given by Example 5.2 and

$$\mathbf{C}=\begin{bmatrix}4.3160 & -2.5771 & -1.4626\\ 2.7122 & 1.8365 & -0.1999\\ 1.3827 & -2.5631 & 4.3419\end{bmatrix}. \tag{6.4}$$

One may easily verify that none of the system matrices are individually real symmetrizable of the first kind. We compute the undamped modal matrices

$$\mathbf{U}=\begin{bmatrix} 0.6498 & 0.7383 & 0.1807 \\ 0.4364 & -0.1676 & -0.8840 \\ 0.6224 & -0.6533 & 0.4311 \end{bmatrix}$$

and

$$\mathbf{V}=\begin{bmatrix} 0.0445 & -0.0108 & 0.990 \\ 0.9245 & -0.3784 & -0.0453 \\ 0.3785 & 0.9256 & -0.0068 \end{bmatrix}$$

and notice that

$$\mathbf{V}^T\mathbf{C}\mathbf{U}=\begin{bmatrix} 3.2290 & 1.3664 & 0.6412 \\ 1.3664 & -2.0229 & 4.4962 \\ 0.6412 & 4.4962 & 2.5404 \end{bmatrix}$$

is a symmetric matrix. So all the conditions outlined in Lemma 6.5 are satisfied. Using the matrix $\mathbf{Q}$ given by (4.3) one has

$$\mathbf{L}=\mathbf{V}\mathbf{Q}=\begin{bmatrix} 0.1735 & 0.2198 & 0.3701 \\ 0.3901 & -0.2469 & -0.1170 \\ -0.8026 & 0.1421 & 0.6215 \end{bmatrix}$$

and

$$\mathbf{R}=\mathbf{U}\mathbf{Q}=\begin{bmatrix} -0.5878 & 0.0988 & 0.6148 \\ 0.0353 & -0.3123 & -0.3683 \\ 0.6926 & -0.1489 & -0.1529 \end{bmatrix}.$$

These real matrices transform the system to a real symmetric form.

7 Concluding Discussions and Further Research

A method of symmetrization of asymmetric linear dynamical systems by means of equivalence transformation is introduced. This new method of symmetrization is called ''symmetrization of the second kind'' and the existing symmetrization method based on the similarity transformation is called ''symmetrization of first kind.'' Because the equivalence transformations are most general linear transformations, symmetrization of the second kind holds that of the first kind as a special case and offers a generalized method to obtain symmetric forms of finite dimensional asymmetric linear systems.

Several results are provided on symmetrization of the second kind. We have discussed real and complex symmetrizability based on the nature of the symmetrizing matrices. It was proved that every rectangular complex matrix is complex symmetrizable of the second kind to a real square matrix. It is possible to perform the symmetrization in such a way that the resulting symmetric form becomes positive definite. As a special case it was shown that all real matrices are real symmetrizable of the second kind. These results are more general than symmetrizability of the first kind which cannot be applied to rectangular matrices. An easy numerical method is presented to calculate symmetric forms of general matrices. This method is much easier than calculating the symmetric form of the first kind using Taussky's factorization.

In view of the undamped and damped dynamical systems, we have discussed simultaneous symmetrizability of two and three real square matrices. It was shown that (truly) undamped asymmetric systems are always real symmetrizable. The condition of real symmetrizability of general undamped systems is derived. We also have given a condition for real symmetrizability of the second kind for damped systems. Our result is more liberal than the existing results and also much easier to check from a numerical point view.

Further research is required to successfully apply the concept of symmetrizability of the second kind in asymmetric linear dynamical systems. By an example it was shown that the condition of symmetrizability of a damped system given by Theorem 6.4 is less restrictive than the existing results. A mathematical proof to establish this fact will be useful. Moreover, this condition is spectral (i.e., an eigensolution calculation is required), a nonspectral condition which is more easy to check needs to be developed. Some of our results (Lemma 5.2 and Lemma 6.5) provide only sufficient conditions for real symmetrizability of systems with general coefficient matrices. More stronger results are required in these directions. Further studies regarding stability of the symmetrizing transformations and the robustness of them with respect to perturbations in the entries of the coefficient matrices are also worth pursuing.

Acknowledgments

Funding provided by the Nehru Memorial Trust, London and the Cambridge Commonwealth Trust is greatly acknowledged.

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