A Non-parametric Approach for Uncertainty Quantification in Structural Dynamics

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Uncertainties are unavoidable in the description of real-life engineering systems. The quantification of uncertainties plays a crucial role in establishing the credibility of a numerical model. Uncertainties can be broadly divided into two categories. The first type is due to the inherent variability in the system parameters, for example, different cars manufactured from a single production line are not exactly the same. This type of uncertainty is often referred to as aleatoric uncertainty. If enough samples are present, it is possible to characterize the variability using well established statistical methods and consequently the probably density functions (pdf) of the parameters can be obtained. The second type of uncertainty is due to the lack of knowledge regarding a system, often referred to as epistemic uncertainty. This kind of uncertainty generally arise in the modelling of complex systems, for example, in the modeling of cabin noise in helicopters. Due to its very nature, it is comparatively difficult to quantify or model this type of uncertainties. In this work a new method based on the random matrix theory and the maximum entropy approach is developed to quantify this type of uncertainties.

Uncertainty Quantification

There are two broad approaches to quantify uncertainties in a model. The first is the parametric approach and the second is the non-parametric approach. In the parametric approach the uncertainties associates with the system parameters, such as Young's modulus, mass density, Poisson's ratio, damping coefficient and geometric parameters are quantified using statistical methods and propagated, for example, using the stochastic finite element method. This type of approach is suitable to quantify aleatoric uncertainties. Epistemic uncertainty on the other hand do not explicitly depend on the systems parameters. For example, there can be unquantified errors associated with the equation of motion (linear on non-linear), in the damping model (viscous or non-viscous), in the model of structural joints, and also in the numerical methods (e.g, discretisation of displacement fields, truncation and roundoff errors, tolerances in the optimization and iterative algorithms, step-sizes in the time-integration methods). It is evident that the parametric approach is not suitable to quantify this type of uncertainties and a non-parametric approach is needed for this purpose. The aim of this work is to develop a general non-parametric uncertainty quantification tool for structural dynamic systems. The proposed approach is based on the random matrix theory and the maximum entropy method. The equation of motion of a damped n-degree-of-freedom linear dynamic system can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices respectively. In order to completely quantify the uncertainties associated with system (1), we need to obtain the probability density functions of the random matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$.

Random Matrix Theory

The probability density function of a random matrix[1–4] can be defined in a manner similar to that of a random variable. If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$. We define three types of random matrices which are relevant to this study.

Definition 1. Gaussian Random Matrix*: The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by*

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} |\mathbf{\Sigma}|^{-p/2} |\mathbf{\Psi}|^{-n/2} \times \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \tag{2}$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Here $\operatorname{etr}\{\bullet\} \equiv \exp\{\operatorname{Tr}(\bullet)\}$ and $|\bullet| \equiv$ determinant of a matrix.

Definition 2. Wishart matrix*: An $n \times n$ random symmetric positive definite matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\mathbf{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by*

$$p_{\mathbf{S}}(\mathbf{S}) = \left\{2^{\frac{1}{2}np}\,\Gamma_n\left(\frac{1}{2}p\right)|\mathbf{\Sigma}|^{\frac{1}{2}p}\right\}^{-1} |\mathbf{S}|^{\frac{1}{2}(p-n-1)} \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}\mathbf{S}\right\} \tag{3}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \mathbf{\Sigma})$.

Definition 3. Matrix Variate Gamma Distribution*: An $n \times n$ random symmetric positive definite matrix $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\mathbf{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by*

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a)\,|\mathbf{\Psi}|^{-a}\right\}^{-1} |\mathbf{W}|^{a-\frac{1}{2}(n+1)} \operatorname{etr}\{-\mathbf{\Psi}\mathbf{W}\}; \quad \Re(a) > \frac{1}{2}(n-1) \tag{4}$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \mathbf{\Psi})$.

In equations (3) and (4), the function $\Gamma_n(a)$ is the multivariate gamma function, which can be expressed as

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \text{ for } \Re(a) > \frac{1}{2}(n-1)$$

Matrix Variate Distribution Using the Maximum Entropy Method

An information theoretic approach is taken[5] to obtain the matrix variate distributions of the random system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. Suppose that the mean values of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given by $\widehat{\mathbf{M}}$, $\widehat{\mathbf{C}}$ and $\widehat{\mathbf{K}}$ respectively. This information is likely to be available, for example, using the deterministic finite element method. However, there are uncertainties associated with our modelling so that $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are actually random matrices. The distribution of these random matrices should be such that they are symmetric, positive-definite and the probability density functions of their inverse matrices should exist. Because the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ have similar probabilistic characteristics, for notational convenience we will use the notation $\mathbf{G}$ which stands for any one the system matrices. Suppose the matrix variate density function of $\mathbf{G} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{G}}(\mathbf{G}) : \mathbb{R}_n^+ \to \mathbb{R}$. We have the following information and constrains to obtain $p_{\mathbf{G}}(\mathbf{G})$:

$$\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = 1 \quad \text{(normalization)} \tag{5}$$

$$\text{and} \quad \int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = \widehat{\mathbf{G}} \quad \text{(the mean matrix)} \tag{6}$$

The mean matrix $\widehat{\mathbf{G}}$ is symmetric and positive definite and the integrals appearing in the above two equations are $n(n+1)/2$ dimensional. To apply the maximum entropy method, first note that the entropy associated with the matrix variate probability density function $p_{\mathbf{G}}(\mathbf{G})$ can be expressed as

$$\mathcal{S}\left(p_{\mathbf{G}}\right) = -\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G}) \ln\left\{p_{\mathbf{G}}(\mathbf{G})\right\} d\mathbf{G} \tag{7}$$

Using this, together with the constrains (5) and (6) we construct the Lagrangian

$$\mathcal{L}\left(p_{\mathbf{G}}\right) = -\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G}) \ln\left\{p_{\mathbf{G}}(\mathbf{G})\right\} d\mathbf{G} + (\lambda_0 - 1) \times \left(\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} - 1\right) + \operatorname{Tr}\left(\mathbf{\Lambda}_1\left[\int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} - \widehat{\mathbf{G}}\right]\right)$$

Here $\lambda_0 \in \mathbb{R}$ and $\mathbf{\Lambda}_1 \in \mathbb{R}_{n,n}$ are the unknown Lagrange multiplies which need to be determined. Using the variational calculus it can be shown that the optimal condition is given by

$$\frac{\partial \mathcal{L}\left(p_{\mathbf{G}}\right)}{\partial p_{\mathbf{G}}} = 0 \quad \text{or} \quad p_{\mathbf{G}}(\mathbf{G}) = \exp\{-\lambda_0\} \operatorname{etr}\{-\mathbf{\Lambda}_1 \mathbf{G}\} \tag{8}$$

Using the matrix calculus[6], the Lagrange multiplies λ_0 and $\mathbf{\Lambda}_1$ can be obtained exactly from equations (5), (6) and (8) to obtain $p_{\mathbf{G}}(\mathbf{G})$. After some algebra it can be shown that

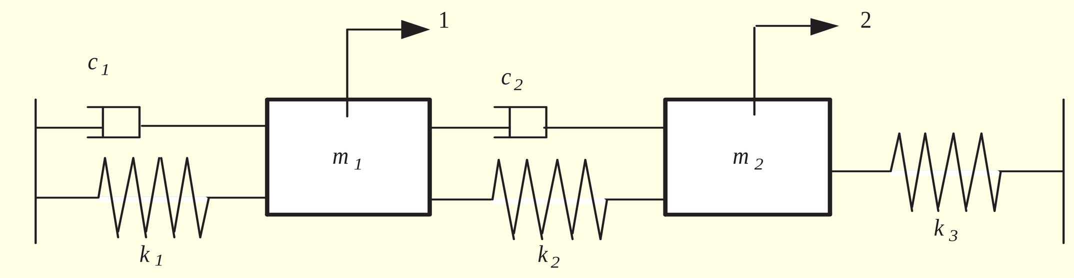
$$p_{\mathbf{G}}(\mathbf{G}) = r^{nr} \left\{\Gamma_n(r)\right\}^{-1} \left|\widehat{\mathbf{G}}\right|^{-r} \operatorname{etr}\left\{-r\widehat{\mathbf{G}}^{-1}\mathbf{G}\right\} \tag{9}$$

where $r = \frac{1}{2}(n+1)$. Comparing equation (9) with the Wishart distribution in equation (3) it can be observed that $\mathbf{G}$ has the Wishart distribution with parameters $p = n+1$ and $\mathbf{\Sigma} = \widehat{\mathbf{G}}/(n+1)$. Therefore, we have the following fundamental result regarding the non-parametric uncertainty modeling of structural dynamic systems:

Theorem 1. *If only the mean of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ is available, say $\widehat{\mathbf{G}}$, then the matrix has a Wishart distribution with parameters $(n+1)$ and $\widehat{\mathbf{G}}/(n+1)$, that is $\mathbf{G} \sim W_n\left(n+1, \widehat{\mathbf{G}}/(n+1)\right)$.*

The pdf of each of the system matrices given by equation (9) depends only on the dimension of the matrix and its mean value. The discovery of the Wishart distribution in this context turns out to be very useful because it has been studied extensively in the literatures of different subjects.

A Simple Example



We consider a simple two-degrees-of-freedom system to illustrate the matrix variate distributions. The mean of the mass, damping and stiffness matrices are given by

$$\widehat{\mathbf{M}} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \widehat{\mathbf{C}} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \quad \widehat{\mathbf{K}} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

For this system $n = 2$ so that $r = \frac{1}{2}(n+1) = 3/2$ and $\Gamma_n(r) = \Gamma_2(3/2) = \pi/2$. Suppose $\mathbf{Z} \in \mathbb{R}_2^+$ is a symmetric positive definite matrix such that $\mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix}$. The positive definite condition requires that $|\mathbf{Z}| > 0$, that is the variables z_{11}, z_{12} and z_{22} should vary such that $z_{11}z_{22} - z_{12}^2 > 0$. Using equation (9), the pdf of the mass matrix can be obtained as

$$p_{\mathbf{M}}(\mathbf{Z}) = \frac{(3/2)^3}{\pi/2} |\widehat{\mathbf{M}}|^{-3/2} \operatorname{etr}\left\{-\frac{3}{2}\widehat{\mathbf{M}}^{-1}\mathbf{Z}\right\} = \frac{27}{4\pi m_1 m_2 \sqrt{m_1 m_2}} \exp\left\{-\frac{3}{2}\frac{z_{11}m_2 + z_{22}m_1}{m_1 m_2}\right\}; \quad z_{11}z_{22} > 0$$

Similarly, the pdfs of the damping and stiffness matrices can be expressed as

$$p_{\mathbf{C}}(\mathbf{Z}) = \frac{27}{4\pi c_1 c_2 \sqrt{c_1 c_2}} \exp\left\{-\frac{3}{2}\frac{z_{11}c_2 + 2z_{12}c_2 + z_{22}(c_1 + c_2)}{c_2 c_1}\right\}$$

and

$$p_{\mathbf{K}}(\mathbf{Z}) = \frac{27}{4\pi\left(k_1k_2 + k_1k_3 + k_2k_3\right)^{3/2}} \times \exp\left\{-\frac{3}{2}\frac{z_{11}(k_2 + k_3) + 2z_{12}k_2 + z_{22}(k_1 + k_2)}{k_1k_2 + k_1k_3 + k_2k_3}\right\}; \quad z_{11}z_{22} - z_{12}^2 > 0$$

Once the pdf of the system matrices are known, one can propagate uncertainties using standard simulation methods.

Conclusions

A new non-parametric method for uncertainty quantification in linear dynamic systems has been proposed. The method is based on the maximum entropy principle and random matrix theory. It is assumed that only the mean of the system matrices are known. The derived probability density function of the random system matrices are completely characterized by the dimension of the matrices and their mean values. The main outcome of this study is that, if only the mean value of a system matrix is known then the matrix follows a Wishart distribution with proper parameters.

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