Random Matrix Models for Structural Dynamics

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Stochastic structural dynamics

- The equation of motion:
  \[ M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = p(t) \]

- Due to the presence of uncertainty \( M, C \) and \( K \) become random matrices.

- The main objectives are:
  - to quantify uncertainties in the system matrices
  - to predict the variability in the response vector \( x \)
Current Methods

Three different approaches are currently available

- **Low frequency**: Stochastic Finite Element Method (SFEM) - considers parametric uncertainties in details
- **High frequency**: Statistical Energy Analysis (SEA) - do not consider parametric uncertainties in details
- **Mid-frequency**: Hybrid method - ‘combination’ of the above two
Random Matrix Method (RMM)

- **The objective**: To have an unified method which will work across the frequency range.

- **The methodology**:
  - Derive the matrix variate probability density functions of $M$, $C$ and $K$
  - Propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)
Outline of the presentation

In what follows next, I will discuss:

- Introduction to Matrix variate distributions
- Maximum entropy distribution
- Optimal Wishart distribution
- Numerical examples
- Open problems & discussions
Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.

- If \( A \) is an \( n \times m \) real random matrix, the matrix variate probability density function of \( A \in \mathbb{R}^{n,m} \), denoted as \( p_A(A) \), is a mapping from the space of \( n \times m \) real matrices to the real line, i.e., \( p_A(A) : \mathbb{R}^{n,m} \rightarrow \mathbb{R} \).
Gaussian random matrix

The random matrix \( X \in \mathbb{R}_{n,p} \) is said to have a matrix variate Gaussian distribution with mean matrix \( M \in \mathbb{R}_{n,p} \) and covariance matrix \( \Sigma \otimes \Psi \), where \( \Sigma \in \mathbb{R}_{n}^{+} \) and \( \Psi \in \mathbb{R}_{p}^{+} \) provided the pdf of \( X \) is given by

\[
p_X (X) = (2\pi)^{-np/2} |\Sigma|^{-p/2} |\Psi|^{-n/2} \exp \left\{ -\frac{1}{2} \Sigma^{-1} (X - M) \Psi^{-1} (X - M)^T \right\}
\]

This distribution is usually denoted as \( X \sim N_{n,p} (M, \Sigma \otimes \Psi) \).
Gaussian orthogonal ensembles

A random matrix $\mathbf{H} \in \mathbb{R}_{n,n}$ belongs to the Gaussian orthogonal ensemble (GOE) provided its pdf of is given by

$$p_{\mathbf{H}}(\mathbf{H}) = \exp\left(-\theta_2 \text{Trace} (\mathbf{H}^2) + \theta_1 \text{Trace} (\mathbf{H}) + \theta_0\right)$$

where $\theta_2$ is real and positive and $\theta_1$ and $\theta_0$ are real. This is a good model for high-frequency vibration problems.
Wishart matrix

An $n \times n$ random symmetric positive definite matrix $S$ is said to have a Wishart distribution with parameters $p \geq n$ and $\Sigma \in \mathbb{R}_+^n$, if its pdf is given by

$$p_S(S) = \left\{ 2^{\frac{1}{2} np} \Gamma_n \left(\frac{1}{2} p\right) |\Sigma|^{\frac{1}{2} p} \right\}^{-1} |S|^{\frac{1}{2} (p-n-1)} \text{etr} \left\{ -\frac{1}{2} \Sigma^{-1} S \right\}$$

This distribution is usually denoted as $S \sim W_n(p, \Sigma)$.

Note: If $p = n + 1$, then the matrix is non-negative definite.
An $n \times n$ random symmetric positive definite matrix $W$ is said to have a matrix variate gamma distribution with parameters $a$ and $\Psi \in \mathbb{R}_n^+$, if its pdf is given by

$$p_W(W) = \left\{ \Gamma_n(a) |\Psi|^{-a} \right\}^{-1} |W|^{a-\frac{1}{2}(n+1)} \text{etr} \{-\Psi W\}; \quad \Re(a) > (n - 1)/2 \quad (3)$$

This distribution is usually denoted as $W \sim G_n(a, \Psi)$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma \left[ a - \frac{1}{2}(k - 1) \right]; \quad \text{for } \Re(a) > (n - 1)/2 \quad (4)$$
The distribution of the random system matrices $M$, $C$ and $K$ should be such that they are

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix

$$D(\omega) = -\omega^2 M + i\omega C + K$$

should exist $\forall \omega$
Distribution of the system matrices

- The exact application of the last constraint requires the derivation of the joint probability density function of $M$, $C$ and $K$, which is quite difficult to obtain.

- We consider a simpler problem where it is required that the inverse moments of each of the system matrices $M$, $C$ and $K$ must exist.

- Provided the system is damped, this will guarantee the existence of the moments of the frequency response function matrix.
Maximum Entropy Distribution

Suppose that the mean values of $M$, $C$ and $K$ are given by $\overline{M}$, $\overline{C}$ and $\overline{K}$ respectively. Using the notation $G$ (which stands for any one the system matrices) the matrix variate density function of $G \in \mathbb{R}_n^+$ is given by $p_G(G): \mathbb{R}_n^+ \rightarrow \mathbb{R}$. We have the following constrains to obtain $p_G(G)$:

\[
\int_{G>0} p_G(G) \, dG = 1 \quad \text{(normalization)} \quad (5)
\]

and

\[
\int_{G>0} G \, p_G(G) \, dG = \overline{G} \quad \text{(the mean matrix)} \quad (6)
\]
Suppose the inverse moments (say up to order $\nu$) of the system matrix exist. This implies that $E \left[ \| G^{-1} \|_F^\nu \right]$ should be finite. Here the Frobenius norm of matrix $A$ is given by

$$\| A \|_F = \left( \text{Trace} (A A^T) \right)^{1/2}.$$  

Taking the logarithm for convenience, the condition for the existence of the inverse moments can be expresses by

$$E \left[ \ln | G |^{-\nu} \right] < \infty$$
The Lagrangian becomes:

\[ \mathcal{L} (p_G) = - \int_{G > 0} p_G (G) \ln \left\{ p_G (G) \right\} \, dG + \]

\[ (\lambda_0 - 1) \left( \int_{G > 0} p_G (G) \, dG - 1 \right) - \nu \int_{G > 0} \ln |G| \, p_G \, dG \]

\[ + \text{Trace} \left( \Lambda_1 \left[ \int_{G > 0} G \, p_G (G) \, dG - \bar{G} \right] \right) \quad (7) \]

Note: \( \nu \) cannot be obtained uniquely!
Using the calculus of variation

\[ \frac{\partial \mathcal{L}(p_G)}{\partial p_G} = 0 \]

or

\[ - \ln \{p_G(G)\} = \lambda_0 + \text{Trace}(\Lambda_1 G) - \ln |G|^\nu \]

or

\[ p_G(G) = \exp\{-\lambda_0\} |G|^\nu \text{etr}\{-\Lambda_1 G\} \]
Using the matrix variate Laplace transform
\((T \in \mathbb{R}_{n,n}, S \in \mathbb{C}_{n,n}, a > (n + 1)/2)\)

\[
\int_{T > 0} \text{etr} \left\{ -ST \right\} |T|^{a-(n+1)/2} dT = \Gamma_n(a) |S|^{-a}
\]

and substituting \(p_G(G)\) into the constraint equations it can be shown that

\[
p_G(G) = r^{-nr} \left\{ \Gamma_n(r) \right\}^{-1} |G|^{-r} |G|^\nu \text{etr} \left\{ -rG^{-1} \right\}
\]

where \(r = \nu + (n + 1)/2.\)
Comparing it with the Wishart distribution we have:

**Theorem 1.** If $\nu$-th order inverse-moment of a system matrix $G \equiv \{M, C, K\}$ exists and only the mean of $G$ is available, say $\overline{G}$, then the maximum-entropy pdf of $G$ follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and $\Sigma = \overline{G}/(2\nu + n + 1)$, that is

$$G \sim W_n \left(2\nu + n + 1, \overline{G}/(2\nu + n + 1)\right).$$
Properties of the Distribution

- Covariance tensor of $G$:

$$\text{cov}(G_{ij}, G_{kl}) = \frac{1}{2\nu + n + 1} (\overline{G}_{ik}\overline{G}_{jl} + \overline{G}_{il}\overline{G}_{jk})$$

- Normalized standard deviation matrix

$$\delta^2_G = \frac{\mathbb{E} \left[ \| G - \mathbb{E}[G] \|_F^2 \right]}{\| \mathbb{E}[G] \|_F^2} = \frac{1}{2\nu + n + 1} \left\{ 1 + \frac{\{\text{Trace}(\overline{G})\}^2}{\text{Trace}(\overline{G}^2)} \right\}$$

- $\delta^2_G \leq \frac{1 + n}{2\nu + n + 1}$ and $\nu \uparrow \Rightarrow \delta^2_G \downarrow$. 
If $G$ is $W_n(p, \Sigma)$ then $V = G^{-1}$ has the inverted Wishart distribution:

$$P_V(V) = \frac{2^{m-n-1}n/2 |\Psi|^{m-n-1}/2}{\Gamma_n[(m - n - 1)/2]} |V|^{m/2} \text{etr} \left\{-\frac{1}{2} V^{-1}\Psi\right\}$$

where $m = n + p + 1$ and $\Psi = \Sigma^{-1}$ (recall that $p = 2\nu + n + 1$ and $\Sigma = \bar{G}/p$)
Mean: $\mathbb{E} \left[ G^{-1} \right] = \frac{pG^{-1}}{p - n - 1}$

cov $\left( G_{ij}^{-1}, G_{kl}^{-1} \right) = \frac{\left(2\nu + n + 1\right)\left(\nu^{-1}G_{ij}^{-1}G_{kl}^{-1} + G_{ik}^{-1}G_{jl}^{-1} + G_{il}^{-1}G_{kj}^{-1}\right)}{2\nu(2\nu + 1)(2\nu - 2)}$
Suppose $n = 101$ & $\nu = 2$. So $p = 2\nu + n + 1 = 106$ and $p - n - 1 = 4$. Therefore, $E[G] = \overline{G}$ and 

$$E[G^{-1}] = \frac{106}{4}\overline{G}^{-1} = 26.5\overline{G}^{-1} \text{ !!!!!!!!!!!}$$

From a practical point of view we do not expect them to be so far apart!

One way to reduce the gap is to increase $p$. But this implies the reduction of variance.
My argument: The distribution of $G$ must be such that $E[G]$ and $E[G^{-1}]$ should be closest to $\overline{G}$ and $\overline{G}^{-1}$ respectively.

Suppose $G \sim W_n \left(n + 1 + \theta, \overline{G}/\alpha\right)$. We need to find $\alpha$ such that the above condition is satisfied.

Therefore, define (and subsequently minimize) ‘normalized errors’:

\[
\varepsilon_1 = \frac{\|\overline{G} - E[G]\|_F}{\|\overline{G}\|_F} \\
\varepsilon_2 = \frac{\|\overline{G}^{-1} - E[G^{-1}]\|_F}{\|\overline{G}^{-1}\|_F}
\]
Because $G \sim W_n (n + 1 + \theta, \bar{G}/\alpha)$ we have

$$E [G] = \frac{n + 1 + \theta}{\alpha} \bar{G}$$

and

$$E [G^{-1}] = \frac{\alpha}{\theta} \bar{G}^{-1}$$

We define the objective function to be minimized as

$$\chi^2 = \varepsilon_1^2 + \varepsilon_2^2 = \left(1 - \frac{n+1+\theta}{\alpha}\right)^2 + \left(1 - \frac{\alpha}{\theta}\right)^2$$
The optimal value of $\alpha$ can be obtained as by setting $\frac{\partial \chi^2}{\partial \alpha} = 0$ or

$$\alpha^4 - \alpha^3 \theta - \theta^4 + (-2n + \alpha - 2) \theta^3 + ((n + 1) \alpha - n^2 - 2n - 1) \theta^2 = 0.$$ 

The only feasible value of $\alpha$ is

$$\alpha = \sqrt{\theta(n + 1 + \theta)}.$$
From this discussion we have the following:

**Theorem 2.** If $\nu$-th order inverse-moment of a system matrix $G \equiv \{M, C, K\}$ exists and only the mean of $G$ is available, say $\overline{G}$, then the unbiased distribution of $G$ follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and

$$
\Sigma = \overline{G} / \sqrt{2\nu(2\nu + n + 1)},
$$

that is

$$
G \sim W_n \left(2\nu + n + 1, \overline{G} / \sqrt{2\nu(2\nu + n + 1)}\right).
$$
Again consider $n = 100$ and $\nu = 2$, so that $\theta = 2\nu = 4$.

In the previous approach $\alpha = 2\nu + n + 1 = 105$. For the optimal distribution, $\alpha = \sqrt{\theta(\theta + n + 1)} = 2\sqrt{105} = 20.49$.

We have $E[G] = \frac{105}{2\sqrt{105}} \overline{G} = 5.12\overline{G}$ and $E[G^{-1}] = \frac{2\sqrt{105}}{4} \overline{G}^{-1} = 5.12\overline{G}^{-1}$.

The overall normalized difference for the previous case is $\chi^2 = 0 + (1 - 105/4)^2 = 637.56$. The same for the optimal distribution is $\chi^2 = 2(1 - \sqrt{105}/2)^2 = 34.01$, which is considerably smaller compared to the non-optimal distribution.
Simulation Algorithm

- Obtain $\theta = \frac{1}{\delta^2 G} \left\{ 1 + \frac{\{\text{Trace}(\bar{G})\}^2}{\text{Trace}(\bar{G}^2)} \right\} - (n + 1)$

- If $\theta < 4$, then select $\theta = 4$.

- Calculate $\alpha = \sqrt{\theta(n + 1 + \theta)}$

- Generate samples of $G \sim W_n(n + 1 + \theta, \bar{G}/\alpha)$
  (MATLAB® command wishrnd can be used to generate the samples)

- Repeat the above steps for all system matrices and solve for every samples
Example: A cantilever Plate

Cantilever plate with a slot: $\mu = 0.3$, $\rho = 8000$ kg/m$^3$, $t = 5$ mm,

$L_x = 2.27$m, $L_y = 1.47$m
Plate Mode 4

Mode 4, freq. = 9.2119 Hz

Fourth Mode shape
Plate Mode 5

Mode 5, freq. = 11.6696 Hz

Fifth Mode shape
Deterministic FRF

FRF of the deterministic plate
Frequency Spacing

Number of modes: 486

Natural frequency spacing distribution (without slot)

Simulation
Rayleigh (Wigner surmise)
exponential

Spacing density $p(s)$

Natural frequency spacing (s), rad/s
Frequency Spacing

Number of modes: 486

Natural frequency spacing distribution (with slot)
Direct finite-element MCS of the amplitude of the cross-FRF of the plate with randomly placed masses; 30 masses, each weighting 0.5% of the total mass of the plate are simulated.
Direct finite-element MCS of the amplitude of the driving-point FRF of the plate with randomly placed masses; 30 masses, each weighting 0.5% of the total mass of the plate are simulated.
MCS of the amplitude of the cross-FRF of the plate using optimal Wishart mass matrix,

\[ n = 429, \delta_M = 2.0449. \]
MCS of the amplitude of the driving-point-FRF of the plate using optimal Wishart mass matrix, $n = 429$, $\delta_M = 2.0449$. 
Comparison of the mean values of the amplitude of the cross-FRF.
Comparison of the mean values of the amplitude of the driving-point-FRF.

Comparison of Mean - 2
Comparison of Variation - 1

Comparison of the 5% and 95% probability points of the amplitude of the cross-FRF.
Comparison of the 5% and 95% probability points of the amplitude of the driving-point-FRF.
Summary & conclusions

- **Wishart matrices** can be used as the distribution for the system matrices in structural dynamics.
- The parameters of the distribution can be obtained by solving an optimisation problem.
Next steps

- Numerical works (validation against??)
- Eigenvalues, eigenvector statistics and calculation of dynamic response.
- Distribution of the dynamic stiffness matrix (complex Wishart matrix?)
- Inversion of the dynamic stiffness matrix (FRFs)
- Distribution of \( \mathbf{Y}(\omega) = \left[ \mathbf{RD}(\omega)^{-1} \mathbf{P} \right] \) where \( \mathbf{P} \in \mathbb{C}_{n,r} \) and \( \mathbf{R} \in \mathbb{R}_{p,n} \)
- Cumulative distribution function of the response (reliability problem)
Open problems & discussions

- Is MEnT appropriate here?
- $\bar{G}$ is just one ‘observation’ - not an ensemble mean.
- What happens if we know the covariance tensor of $G$ (e.g., using Stochastic Finite element Method)?
- What if the zeros in $G$ are not preserved?
Structure of the Matrices

Nonzero elements of the system matrices