Identification of Damping Using Proper Orthogonal Decomposition

M Khalil, S Adhikari and A Sarkar

Department of Aerospace Engineering, University of Bristol, Bristol, U.K.
Email: S.Adhikari@bristol.ac.uk
Outline

- Motivation
- Brief overview of damping identification
- Independent Component Analysis
- Numerical Validation
- Conclusions
Unlike the inertia and stiffness forcers, in general damping cannot be obtained using ‘first principle’. Two broad approaches are:

- damping identification from modal testing and analysis
- direct damping identification from the forced response measurements in the frequency or time domain


Some shortcomings of the modal analysis based methodologies are:

- Difficult to extend in the mid-frequency range: Relies on the presence of FRF distinct peaks
- Computationally expensive and time-consuming for large systems
- Non-proportional damping leading to complex modes adds to the computational burden
Mid-Frequency Range

- Low-Frequency Range: A uniform low modal density
- High-Frequency Range: A uniform high modal density
- Mid-Frequency Range: Intermediate band in which modal density varies greatly
The equations describing the forced vibration of a viscously damped linear discrete system with \( n \) dof:

\[
M_n \ddot{u}_n(t) + C_n \dot{u}_n(t) + K_n u_n(t) = f_n(t)
\]

- \( M_n \) is the mass matrix, \( C_n \) is the damping matrix and \( K_n \) is the stiffness matrix.
- \( u_n(t) \) is the displacement vector, and \( f_n(t) \) is the forcing vector at time \( t \).
- In the frequency domain, one has

\[
[-\omega^2 M_n + i\omega C_n + K_n] U_n(\omega) = F_n(\omega)
\]
Damping Matrix Identification

- Applying Kronecker Algebra and taking the vec operator to the frequency domain representation

\[(U_n(\omega_i)^T \otimes i\omega_i I_n) \text{vec} C_n = F_n(\omega_i) + \omega_i^2 M_n U_n(\omega_i) - K_n U_n(\omega_i),\]

- For many frequencies, we have

\[\begin{bmatrix}
U_n(\omega_1)^T \otimes i\omega_1 I_n \\
U_n(\omega_2)^T \otimes i\omega_2 I_n \\
\vdots \\
U_n(\omega_J)^T \otimes i\omega_J I_n
\end{bmatrix} \text{vec} C_n = \begin{bmatrix}
F_n(\omega_1) + \omega_1^2 M_n U_n(\omega_1) - K_n U_n(\omega_1) \\
F_n(\omega_2) + \omega_2^2 M_n U_n(\omega_2) - K_n U_n(\omega_2) \\
\vdots \\
F_n(\omega_J) + \omega_J^2 M_n U_n(\omega_J) - K_n U_n(\omega_J)
\end{bmatrix}.\]

- The above equation can be written as

\[Ax = y\]
Least-Square Approach

- In case the system of equations being overdetermined, $\mathbf{x}$ can be solved in the least-square sense using the least-square inverse of the matrix $\mathbf{A}$, as follows

$$\hat{\mathbf{x}} = [\mathbf{A}^T \mathbf{A}]^{-1} \mathbf{A}^T \mathbf{y}.$$ 

- $\hat{\mathbf{x}}$ is the least-square estimate of $\mathbf{x}$ and $[\mathbf{A}^T \mathbf{A}]^{-1} \mathbf{A}^T$ is the Moore-Penrose inverse of $\mathbf{A}$. 

Physics-Based Tikhonov Regularisation

- In order to satisfy symmetry, for instance, in the damping matrix $C_m$, we need to have

$$C_n = C_n^T$$

- The symmetry condition in the mass matrix gives rise to the constraint equation:

$$L_C x = 0_{n^2}$$

- $0_{m^2}$ is the zero vector of order $m^2$ and the subscript in $L_C$ indicates that the constraint is on the damping matrix.
Applying Tikhonov Regularisation to estimate $x$, we obtain the following solution

$$
\hat{x} = \left( A^T A + \lambda_C^2 L_C^T L_C \right)^{-1} (A^T y).
$$

The above solution depends on the values chosen for the regularisation parameter $\lambda_C$

If $\lambda_C$ is very large, the constraint enforcing the symmetry condition predominates in the solution of $x$

On the other hand, if it is chosen to be small, the symmetry constraint is less satisfied and the solution depends more heavily on the observed data $y$
The Need for Model Order Reduction

- In the proposed recursive least squares method, we are required to obtain the inverse of a square matrix of order $n^2$.
- If we are trying to estimate the damping matrix of a complex system with large $n$, this is not feasible, even on high performance computers.
- There is a need to reduce the order of the model prior to the system identification step.
Proper Orthogonal Decomposition

- Entails the extraction of the dominant eigenspace of the response correlation matrix over a given frequency band
- These dominant eigenvectors span the system response optimally on the prescribed frequency range of interest
- POD is essentially the following eigenvalue problem

\[ R_{uu} \varphi = \lambda \varphi \]

- \( R_{uu} \) is the response correlation matrix given by

\[ R_{uu} = \left\langle u_n(t) u_n(t)^T \right\rangle \approx \frac{1}{T} \sum_{t=1}^{T} u_n(t) u_n^T(t) \]
Spectral Decomposition of $R_{uu}$

- Using the spectral decomposition of $R_{uu}$, one obtains

$$R_{uu} = \sum_{i=1}^{n} \lambda_i \phi_i \phi_i^T$$

- The eigenvalues are arranged: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$

- The first few modes capture most of the system's energy, i.e. $R_{uu}$ can be approximated by

$$R_{uu} \approx \sum_{i=1}^{m} \lambda_i \phi_i \phi_i^T$$

- $m$ is the number of dominant POD modes, generally much smaller than $n$
The output vector can be approximated by a linear representation involving the first $m$ POD modes using

$$u_n(t) = \sum_{i=1}^{m} a_i(t) \varphi_i = \Sigma a(t)$$

$\Sigma$ is the matrix containing the first $m$ dominant POD eigenvectors:

$$\Sigma = [\varphi_1, \ldots, \varphi_m] \in \mathbb{R}^{n \times m}$$
Model Reduction using POD

Using $\Sigma$ as a transformation matrix, our reduced order model becomes

$$\Sigma^T M_n \Sigma \ddot{\mathbf{a}}(t) + \Sigma^T C_n \Sigma \dot{\mathbf{a}}(t) + \Sigma^T K_n \Sigma \mathbf{a}(t) = \Sigma^T f_n(t)$$

The system of equations can now be rewritten in the reduced-order dimension as

$$M_m \ddot{\mathbf{u}}_m(t) + C_m \dot{\mathbf{u}}_m(t) + K_m \mathbf{u}_m(t) = f_m(t)$$
Model Reduction using POD

- The reduced order mass, damping, and stiffness matrices as well as the reduced order displacement and forcing vectors are

\[
M_m = \Sigma^T M_n \Sigma \in \mathbb{R}^{m \times m} \\
C_m = \Sigma^T C_n \Sigma \in \mathbb{R}^{m \times m} \\
K_m = \Sigma^T K_n \Sigma \in \mathbb{R}^{m \times m} \\
u_m(t) = \Sigma^T u_n(t) = a(t) \\
f_m(t) = \Sigma^T f_f(t)
\]
Reduced-Order Model Identification

- Once either the POD transformation is applied, there will be $m^2$ unknowns to be identified, as opposed to $n^2$ for our original model, where $m$ is much smaller than $n$.

- The aforementioned least square estimation method can now be used to estimate the reduced order damping matrix.

- Once the reduced order damping matrix is estimated, we can carry out system simulations at the lower order dimension $m$, and project the displacement results back into the original $n$-dimensional space.
Numerical Validation

- A coupled linear array of mass-spring oscillators is considered to be the original system.
- A lighter system is coupled with a heavier system.
- The lighter system possesses higher modal densities compared to the heavier system.
System Description

- The mass and stiffness matrices have the form

\[
M_n = \begin{bmatrix} m_1 I_{n/2} & 0_{n/2} \\ 0_{n/2} & m_2 I_{n/2} \end{bmatrix}, \quad K_n = k_u \begin{bmatrix} 2 & -1 \\ -1 & 2 & -1 \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 & -1 \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}
\]

- \( n, m_1, \) and \( m_2 \) are chosen to be 100 DOFs, 0.1 kg, and 1 kg and \( k_u = 4 \times 10^5 \) N/m

- The system is assumed to have Rayleigh damping by \( C_n = \alpha_0 M_n + \alpha_1 K_n \), where \( \alpha_0 = 0.5 \) and \( \alpha_1 = 3 \times 10^{-5} \)
The frequency range considered for the construction of the POD is shown.
POD Eigenvalues

Normalized eigenvalues of the correlation matrix

Log (dB) of normalized POD eigenvalues ($\lambda / \lambda_{max}$)

Eigenvalue number
A typical FRF of the POD reduced model is compared with the original FRF below.

The reduced order model FRF match reasonably well with the original FRF in the frequency band of interest.
Noise-Free Identification

- In the noise-free case, we obtain the identified POD reduced order damping matrix.

- The identified matrix is used to obtain a typical FRF of the system below.

\[ H(\omega) \]

Graph showing FRF comparison between original and POD-identified TFs.
Effect of Noise

- The system response is contaminated with noise
- The variance of the noise is ten times smaller than that of the response
- We obtain the identified FRF shown below
Conclusion

The salient features that emerged from the current investigation are:

- POD can be successfully applied for reduced-order modelling
- Kronecker Algebra in conjunction with Tikhonov Regularisation provide an elegant theoretical formulation involving identification of the damping matrix
- Using a noise-sensitivity study, the identification method is demonstrated to be robust in a noisy environment