Uncertainty Quantification in Structural Dynamics: A Reduced Random Matrix Approach

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Outline of the presentation

- Introduction: current status and challenges
- Uncertainty Propagation (UP) in structural dynamics
  - Parametric uncertainty
  - Nonparametric uncertainty
- Reduced Wishart random matrix model
  - Analytical derivation
  - Parameter estimation
- Computational results
- Experimental results
- Integration with commercial Finite Element code
- Conclusions & future directions
Ensembles of structural dynamical systems

Many structural dynamic systems are manufactured in a production line (nominally identical systems)
A complex structural dynamical system

Complex aerospace system can have millions of degrees of freedom and significant ‘errors’ and/or ‘lack of knowledge’ in its numerical (Finite Element) model.
Sources of uncertainty

(a) **parametric uncertainty** - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) **model inadequacy** - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) **experimental error** - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(d) **computational uncertainty** - e.g., machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis, and
(e) **model uncertainty** - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.
The main difficulties are:

- the **computational time** can be prohibitively high compared to a deterministic analysis for real problems,
- the **volume of input data** can be unrealistic to obtain for a credible probabilistic analysis,
- the **predictive accuracy** can be poor if considerable resources are not spent on the previous two items, and
Two different approaches are currently available

- **Parametric approaches**: Such as the **Stochastic Finite Element Method (SFEM)**:
  - aim to characterize parametric uncertainty (type ‘a’)
  - assume that stochastic fields describing parametric uncertainties are known in details
  - suitable for low-frequency dynamic applications (building under earthquake load)
Nonparametric approaches: Such as the Random matrix theory:

- aim to characterize nonparametric uncertainty (types ‘b’ - ‘e’)
- does not consider parametric uncertainties in details
- suitable for high/mid-frequency dynamic applications (e.g., noise propagation in vehicles)
Dynamics of a general linear system

The equation of motion:

$$\ddot{M}\dot{q}(t) + C\dot{q}(t) + Kq(t) = f(t)$$  \hspace{1cm} (1)

- Due to the presence of uncertainty $M$, $C$ and $K$ become random matrices.
- The main objectives in the ‘forward problem’ are:
  - to quantify uncertainties in the system matrices (and consequently in the eigensolutions)
  - to predict the variability in the response vector $q$
Random matrix model for dynamical system

Suppose $H(x, \theta)$ is a distributed random field describing a system parameter. This can be expanded using the Karhunen-Loève expansion as

$$H(x, \theta) = H_0(x) + \epsilon \sum_{j=1}^{M} \xi_j(\theta) \sqrt{\lambda_j} \varphi_j(x)$$  \hspace{1cm} (2)

where $H_0(x)$ is the mean of the random field, $\epsilon$ is its standard deviation and $M$ is the number of terms used to truncate the infinite series.

Substituting this in the equation of motion and following the usual finite element method, and of the system matrix can be expressed as

$$G(\theta) = G_0 + \epsilon_G \sum_{j=1}^{M} \xi_{G_j}(\theta) G_j$$  \hspace{1cm} (3)

Q: how non parametric uncertainties can be taken into account?
The probability density function of a random matrix can be defined in a manner similar to that of a random variable.

If $A$ is an $n \times m$ real random matrix, the matrix variate probability density function of $A \in \mathbb{R}^{n,m}$, denoted as $p_A(A)$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_A(A) : \mathbb{R}^{n,m} \rightarrow \mathbb{R}$. 
The random matrix $X \in \mathbb{R}^{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $M \in \mathbb{R}^{n,p}$ and covariance matrix $\Sigma \otimes \Psi$, where $\Sigma \in \mathbb{R}^+_n$ and $\Psi \in \mathbb{R}^+_p$ provided the pdf of $X$ is given by

$$p_X(X) = (2\pi)^{-np/2} |\Sigma|^{-p/2} |\Psi|^{-n/2}$$

$$\text{etr} \left\{ -\frac{1}{2} \Sigma^{-1}(X - M) \Psi^{-1}(X - M)^T \right\}$$

(4)

This distribution is usually denoted as $X \sim N_{n,p}(M, \Sigma \otimes \Psi)$. 
Matrix variate Gamma distribution

A $n \times n$ symmetric positive definite matrix random $W$ is said to have a matrix variate gamma distribution with parameters $a$ and $\Psi \in \mathbb{R}_n^+$, if its pdf is given by

$$p_W(W) = \left\{ \Gamma_n(a) |\Psi|^{-a} \right\}^{-1} |W|^{a-\frac{1}{2}(n+1)} \; \text{etr} \left\{ -\Psi W \right\}; \; \Re(a) > \frac{1}{2}(n-1)$$

This distribution is usually denoted as $W \sim G_n(a, \Psi)$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma \left[ a - \frac{1}{2}(k - 1) \right]; \; \text{for } \Re(a) > (n - 1)/2$$
A $n \times n$ symmetric positive definite random matrix $S$ is said to have a Wishart distribution with parameters $p \geq n$ and $\Sigma \in \mathbb{R}^+_n$, if its pdf is given by

$$p_S(S) = \left\{2^{\frac{1}{2}np} \Gamma_n \left(\frac{1}{2}p\right) |\Sigma|^{-\frac{1}{2}p}\right\}^{-1} |S|^{-\frac{1}{2}(p-n-1)} \text{etr} \left\{-\frac{1}{2} \Sigma^{-1}S \right\}$$

(5)

This distribution is usually denoted as $S \sim W_n(p, \Sigma)$. 

Suppose that the mean values of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given by $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$ respectively. Using the notation $\mathbf{G}$ (which stands for any one the system matrices) the matrix variate density function of $\mathbf{G} \in \mathbb{R}^+_n$ is given by $p_{\mathbf{G}}(\mathbf{G}) : \mathbb{R}^+_n \rightarrow \mathbb{R}$. We have the following constrains to obtain $p_{\mathbf{G}}(\mathbf{G})$:

\[
\int_{\mathbf{G} > 0} p_{\mathbf{G}}(\mathbf{G}) \, d\mathbf{G} = 1 \quad \text{(normalization)} 
\]

\[
\text{and} \quad \int_{\mathbf{G} > 0} \mathbf{G} \, p_{\mathbf{G}}(\mathbf{G}) \, d\mathbf{G} = \overline{\mathbf{G}} \quad \text{(the mean matrix)} 
\]
Further constraints

- Suppose that the inverse moments up to order $\nu$ of the system matrix exist. This implies that $E \left[ \| G^{-1} \|_F^{\nu} \right]$ should be finite. Here the Frobenius norm of matrix $A$ is given by

$$\| A \|_F = \left( \text{Trace} \left( A A^T \right) \right)^{1/2}.$$ 

- Taking the logarithm for convenience, the condition for the existence of the inverse moments can be expresses by

$$E \left[ \ln | G |^{-\nu} \right] < \infty$$
The random matrix model

- Following the maximum entropy method it can be shown that the system matrices are distributed as Wishart matrices, i.e.,
  \[ G \sim W_n(G_0, \delta_G^2) \]
- Here \( G_0 \) is the mean and the dispersion parameter (normalized) standard deviation of the system matrices:
  \[
  \delta_G^2 = \frac{\text{E} \left[ \| G - \text{E} [G] \|_F^2 \right]}{\| \text{E} [G] \|_F^2}.
  \] (8)
- This method is computationally expensive as the simulation of two Wishart matrices and the solution of a generalized eigenvalue problem is necessary for each sample.
The dispersion parameter

\[ \delta^2_G = \frac{E \left[ \left\| \epsilon_G \sum_{j=1}^M \xi_{G_j}(\theta)G_j \right\|^2_F \right]}{\|E[G]\|_F^2} \]  

(9)

Since both trace and expectation operators are linear they can be swapped. Doing this we obtain

\[ \delta^2_G = \frac{\epsilon^2_G \text{Trace} \left( E \left[ (\sum_{j=1}^M \sum_{k=1}^M \xi_{G_j}(\theta)\xi_{G_k}(\theta)G_jG_k) \right] \right)}{\|G_0\|_F^2} \]  

(10)

Recalling that the matrices \( G_j \) are not random and \( \{\xi_{G_1}(\theta), \xi_{G_2}(\theta), \ldots\} \) is a set of uncorrelated random variables with zero mean and \( E \left[ \xi_{G_j}(\theta)\xi_{G_k}(\theta) \right] = \delta_{jk} \), we have

\[ \delta^2_G = \frac{\epsilon^2_G \text{Trace} \left( (\sum_{j=1}^M \sum_{k=1}^M E \left[ \xi_{G_j}(\theta)\xi_{G_k}(\theta) \right] G_jG_k) \right)}{\|G_0\|_F^2} \]  

(11)

\[ = \frac{\epsilon^2_G \text{Trace} \left( (\sum_{j=1}^M G_j^2) \right)}{\|G_0\|_F^2} = \frac{\epsilon^2_G \sum_{j=1}^M \|G_j\|^2_F}{\|G_0\|_F^2} \]
Taking the Laplace transform of the equation of motion:

\[
[s^2 \mathbf{M} + s \mathbf{C} + \mathbf{K}] \bar{\mathbf{q}}(s) = \bar{\mathbf{f}}(s)
\]  

The aim here is to obtain the statistical properties of \( \bar{\mathbf{q}}(s) \in \mathbb{C}^n \) when the system matrices are random matrices.

The system eigenvalue problem is given by

\[
\mathbf{K} \phi_j = \omega_j^2 \mathbf{M} \phi_j, \quad j = 1, 2, \ldots, n
\]  

where \( \omega_j^2 \) and \( \phi_j \) are respectively the eigenvalues and mass-normalized eigenvectors of the system.

Suppose the number of modes to be retained is \( m \). In general \( m \ll n \). We form the truncated undamped modal matrices

\[
\mathbf{\Omega} = \text{diag} [\omega_1, \omega_2, \ldots, \omega_m] \in \mathbb{R}^{m \times m} \quad \text{and} \quad \Phi = [\phi_1, \phi_2, \ldots, \phi_m] \in \mathbb{R}^{n \times m}
\]

so that \( \Phi^T \mathbf{K}_e \Phi = \mathbf{\Omega}^2 \) and \( \Phi^T \mathbf{M} \Phi = \mathbf{I}_m \)
Reduced random matrix approach (2)

- Transforming it into the modal coordinates:

\[
[s^2I_m + sC' + \Omega^2] \bar{q}' = \bar{f}'
\]  

(15)

- Here

\[
C' = \Phi^T C \Phi = 2\zeta \Omega, \quad \bar{q} = \Phi \bar{q}' \quad \text{and} \quad \bar{f}' = \Phi^T \bar{f}
\]

(16)

- When we consider random systems, the matrix of eigenvalues \(\Omega^2\) will be a random matrix of dimension \(m\). Suppose this random matrix is denoted by \(\Xi \in \mathbb{R}^{m \times m} : \)

\[
\Omega^2 \sim \Xi
\]

(17)
Since $\Xi$ is a symmetric and positive definite matrix, it can be diagonalized by an orthogonal matrix $\Psi_r$ such that

$$\Psi_r^T \Xi \Psi_r = \Omega_r^2$$  \hspace{1cm} (18)

Here the subscript $r$ denotes the random nature of the eigenvalues and eigenvectors of the random matrix $\Xi$.

Recalling that $\Psi_r^T \Psi_r = I_m$ we obtain

$$\bar{q}' = \left[ s^2 I_m + sC' + \Omega^2 \right]^{-1} \bar{f}'$$ \hspace{1cm} (19)

$$= \Psi_r \left[ s^2 I_m + 2s\zeta \Omega_r + \Omega_r^2 \right]^{-1} \Psi_r^T \bar{f}'$$ \hspace{1cm} (20)
The response in the original coordinate can be obtained as

$$\bar{q}(s) = \Phi \bar{q}'(s) = \Phi \Psi_r \left[ s^2 I_m + 2s \zeta \Omega_r + \Omega_r^2 \right]^{-1} (\Phi \Psi_r)^T \bar{f}(s)$$

$$= \sum_{j=1}^{m} \frac{x_{rj}^T \bar{f}(s)}{s^2 + 2s \zeta_j \omega_{rj} + \omega_{rj}^2} x_{rj}.$$ 

Here

$$\Omega_r = \text{diag} [\omega_{r1}, \omega_{r2}, \ldots, \omega_{rm}] , \quad X_r = \Phi \Psi_r = [x_{r1}, x_{r2}, \ldots, x_{rm}]$$

are respectively the matrices containing random eigenvalues and eigenvectors of the system.
Wishart system matrices

\( M \) and \( K \) are Wishart matrices. For this case \( M \sim W_n(p_1, \Sigma_1) \), \( K \sim W_n(p_1, \Sigma_1) \) with \( E[M] = M_0 \) and \( E[M] = M_0 \).

Here

\[
\Sigma_1 = M_0/p_1, \quad p_1 = \frac{\gamma_M + 1}{\delta^2_M}
\]  \hspace{1cm} (21)

and

\[
\Sigma_2 = K_0/p_2, \quad p_2 = \frac{\gamma_K + 1}{\delta^2_K}
\]  \hspace{1cm} (22)

\[
\gamma_G = \frac{\{\text{Trace} \left( G_0 \right)\}^2}{\text{Trace} \left( G_0^2 \right)}
\]  \hspace{1cm} (23)
We have $\Xi \sim W_m(p, \Omega_0^2/\theta)$ with $E[\Xi^{-1}] = \Omega_0^{-2}$ and $\delta_{\Xi} = \delta_H$. This requires the simulation of one $n \times n$ uncorrelated Wishart matrix and the solution of an $n \times n$ standard eigenvalue problem. The parameters can be obtained as

$$p = n + 1 + \theta \quad \text{and} \quad \theta = \frac{(1 + \gamma_H)}{\delta_H^2} - (n + 1) \quad (24)$$
Defining $\mathbf{H}_0 = \mathbf{M}_0^{-1} \mathbf{K}_0$, the constant $\gamma_H$:

$$
\gamma_H = \frac{\{\text{Trace} (\mathbf{H}_0)\}^2}{\text{Trace} (\mathbf{H}_0^2)} = \frac{\{\text{Trace} (\Omega_0^2)\}^2}{\text{Trace} (\Omega_0^4)} = \frac{\left(\sum_j \omega_{0j}^2\right)^2}{\sum_j \omega_{0j}^4}
$$

(25)

Obtain the dispersion parameter of the generalized Wishart matrix

$$
\delta_H = \frac{\left(p_1^2 + (p_2 - 2 - 2n) p_1 + (-n - 1) p_2 + n^2 + 1 + 2n\right) \gamma_H}{p_2 (-p_1 + n)(-p_1 + n + 3)}
$$

$$
+ \frac{p_1^2 + (p_2 - 2n) p_1 + (1-n) p_2 - 1 + n^2}{p_2 (-p_1 + n)(-p_1 + n + 3)}
$$

(26)
Computational strategy (1)

- Calculate the parameters

\[ \theta = \frac{(1 + \beta_H)}{\delta_H^2} - (m + 1) \quad \text{and} \quad p = [m + 1 + \theta] \tag{27} \]

where \( p \) is approximated to the nearest integer of \( m + 1 + \theta \).

- Create an \( m \times p \) matrix \( Y \) such that

\[ Y_{ij} = \omega_0_i \frac{\hat{Y}_{ij}}{\sqrt{\theta}}; \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, p \tag{28} \]

where \( \hat{Y}_{ij} \) are independent and identically distributed (i.i.d.) Gaussian random numbers with zero mean and unit standard deviation.

- Simulate the \( m \times m \) Wishart random matrix

\[ \Xi = YY^T \quad \text{or} \quad \Xi_{ij} = \frac{\omega_0_i \omega_0_j}{\theta} \sum_{k=1}^{p} \hat{Y}_{ik} \hat{Y}_{jk}; \quad i = 1, 2, \ldots, m; j = 1, 2, \ldots, m \tag{29} \]

Since \( \Xi \) is symmetric, only the upper or lower triangular part need to be simulated.
Computational strategy (2)

- Solve the symmetric eigenvalue problem \((\Omega_r, \Psi_r \in \mathbb{R}^{m \times m})\) for every sample

\[
\Xi \Psi_r = \Omega_r^2 \Psi_r
\]  

(30)

and obtain the random eigenvector matrix

\[
X_r = \Phi_0 \Psi_r = [x_{r1}, x_{r2}, \ldots, x_{rm}] \in \mathbb{R}^{n \times m}
\]  

(31)

- Finally calculate the dynamic response in the frequency domain as

\[
\bar{q}_r(i\omega) = \sum_{j=1}^{m} \frac{x_{rj}^T \bar{f}(s)}{-\omega^2 + 2i\omega\zeta_j \omega_r + \omega_r^2} x_{rj}
\]  

(32)

- The samples of the response in the time domain can also be obtained from the random eigensolutions as

\[
q_r(t) = \sum_{j=1}^{m} a_{rj}(t)x_{rj}, \text{ where } a_{rj}(t) = \frac{1}{\omega_{rj}} \int_0^t x_{rj}^T f(\tau) e^{-\zeta_j \omega_r (t - \tau)} \sin (\omega_{rj} (t - \tau)) d\tau
\]  

(33)
Numerical Examples
A vibrating cantilever plate

Baseline Model: Thin plate elements with 0.7% modal damping assumed for all the modes.
Physical properties

<table>
<thead>
<tr>
<th>Plate Properties</th>
<th>Numerical values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ((L_x))</td>
<td>998 mm</td>
</tr>
<tr>
<td>Width ((L_y))</td>
<td>530 mm</td>
</tr>
<tr>
<td>Thickness ((t_h))</td>
<td>3.0 mm</td>
</tr>
<tr>
<td>Mass density ((\rho))</td>
<td>7860 kg/m(^3)</td>
</tr>
<tr>
<td>Young’s modulus ((E))</td>
<td>2.0 (\times) 10(^5) MPa</td>
</tr>
<tr>
<td>Poisson’s ratio ((\mu))</td>
<td>0.3</td>
</tr>
<tr>
<td>Total weight</td>
<td>12.47 kg</td>
</tr>
</tbody>
</table>

Material and geometric properties of the cantilever plate considered for the experiment. The data presented here are available from http://engweb.swan.ac.uk/~adhikaris/uq/.
Uncertainty type 1: random fields

The Young’s modulus, Poissons ratio, mass density and thickness are random fields of the form

\[
E(x) = \bar{E} (1 + \epsilon_E f_1(x))
\] (34)

\[
\mu(x) = \bar{\mu} (1 + \epsilon_\mu f_2(x))
\] (35)

\[
\rho(x) = \bar{\rho} (1 + \epsilon_\rho f_3(x))
\] (36)

\[
t(x) = \bar{t} (1 + \epsilon_t f_4(x))
\] (37)

- The strength parameters: \( \epsilon_E = 0.15, \epsilon_\mu = 0.15, \epsilon_\rho = 0.10 \) and \( \epsilon_t = 0.15 \).

- The random fields \( f_i(x), i = 1, \cdots, 4 \) are delta-correlated homogenous Gaussian random fields.
Here we consider that the baseline plate is ‘perturbed’ by attaching 10 oscillators with random spring stiffnesses at random locations.

This is aimed at modeling non-parametric uncertainty.

This case will be investigated experimentally.
Methodologies compared

- **Method 1 - Mass and stiffness matrices are fully correlated Wishart matrices:** For this case \( M \sim W_n(p_M, \Sigma_M) \), \( K \sim W_n(p_K, \Sigma_K) \) with \( E[M] = M_0 \) and \( E[M] = M_0 \). This method requires the simulation of two \( n \times n \) fully correlated Wishart matrices and the solution of a \( n \times n \) generalized eigenvalue problem with two fully populated matrices. The computational cost of this approach is \( \approx 2O(n^3) \).

- **Method 2 - Generalized Wishart Matrix:** For this case \( \Xi \sim W_n(p, \Omega_0^2/\theta) \) with \( E[\Xi^{-1}] = \Omega_0^{-2} \) and \( \delta_\Xi = \delta_H \). This requires the simulation of one \( n \times n \) uncorrelated Wishart matrix and the solution of an \( n \times n \) standard eigenvalue problem. The computational cost of this approach is \( \approx O(n^3) \).

- **Method 3 - Reduced diagonal Wishart Matrix:** For this case \( \tilde{\Xi} \sim W_m(\tilde{p}, \tilde{\Omega}_0^2/\theta) \) with \( E[\tilde{\Xi}^{-1}] = \tilde{\Omega}_0^{-2} \) and \( \delta_{\tilde{\Xi}} = \delta_H \). This requires the simulation of one \( m \times m \) uncorrelated Wishart matrix and the solution of a \( m \times m \) standard eigenvalue problem. For large complex systems \( m \) can be significantly smaller than \( n \). The computational cost of this approach is \( \approx O(m^3) \).
Mean of the amplitude of the response of the cross-FRF of the plate, \( n = 1200 \), \( \sigma_M = 0.078 \) and \( \sigma_K = 0.205 \).

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Mean of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$. 
Standard deviation of the amplitude of the response of the cross-FRF of the plate, \( n = 1200, \sigma_M = 0.078 \) and \( \sigma_K = 0.205 \).
Standard deviation of the amplitude of the response of the driving-point-FRF of the plate, $n = 1200$, $\sigma_M = 0.078$ and $\sigma_K = 0.205$. 
Experimental investigation for uncertainty type 2 (randomly attached oscillators)
A cantilever plate: front view

A cantilever plate: side view

The test rig for the cantilever plate; side view.
Comparison of the mean of the amplitude obtained using the experiment and the reduced Wishart matrix approach for the plate with randomly attached oscillators.
Comparison of the mean of the amplitude obtained using the experiment and the reduced Wishart matrix approach for the plate with randomly attached oscillators.
Comparison of relative standard deviation of the amplitude obtained using the experiment and the reduced Wishart matrix approach for the plate with randomly attached oscillators.
Comparison of relative standard deviation of the amplitude obtained using the experiment and the reduced Wishart matrix approach for the plate with randomly attached oscillators.
Integration with ANSYS™

The Finite Element (FE) model of an aircraft wing (5907 degrees-of-freedom). The width is 1.5m, length is 20.0m and the height of the aerofoil section is 0.3m. The material properties are: Young’s modulus 262Mpa, Poisson’s ratio 0.3 and mass density 888.10kg/m³. Input node number: 407 and the output node number 96. A 2% modal damping factor is assumed for all modes.
Vibration modes

Mode 3, frequency 19.047Hz, Mode 5, frequency 53.628Hz

Mode 10, frequency 168.249Hz, Mode 20, frequency 403.711Hz
Mean of a Cross-FRF

Baseline and mean of the amplitude of a cross FRF obtained using the proposed reduced approach for the four sets of dispersion parameters.
Standard deviation of a Cross-FRF

Standard deviation of the amplitude of a cross FRF obtained using the proposed reduced approach for the four sets of dispersion parameters.
Conclusions

- Linear multiple degrees of freedom dynamic systems with uncertain properties are considered.
- A general uncertain propagation approach based on reduced Wishart random matrix is discussed and the results are compared with experimental results.
- Based on numerical and experimental studies, a suitable simple Wishart random matrix model has been identified and a simulation based computational method has been proposed.
- The proposed reduced approach has been integrated with a commercial FE software.