On the dynamics of a Duffing oscillator with an exponential non-viscous damping model

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Linear non-viscous system

The equation of motion:

\[ m \ddot{u}(t) + \int_{0}^{t} c e^{-\mu(t-\tau)} \dot{u}(\tau) \, d\tau + k u(t) = f(t) \]  

(1)
\[
\bar{d}(s) \bar{u}(s) = \bar{p}(s) \tag{2}
\]

where

\[
\bar{d}(s) = s^2 + s \cdot 2\zeta \omega_n \left( \frac{\omega_n}{s\beta + \omega_n} \right) + \omega_n^2 \tag{3}
\]

\(\bar{p}(s)\) is the equivalent forcing function and

\[
\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2\sqrt{km}}, \quad \text{and} \quad \beta = \frac{\omega_n}{\mu}. \tag{4}
\]

\(\omega_n\): undamped natural frequency, \(\zeta\): viscous damping factor and \(\beta\): non-viscous damping factor.
Conditions for oscillatory motion

Critical values of $\zeta$ and $\beta$ for oscillatory (periodic) motion.

$\beta_c = \frac{1}{3(3)^{1/2}}$

$\zeta_c = \frac{4}{3(3)^{1/2}}$

Non-viscous damping factor: $\beta$

Viscous damping factor: $\zeta$

$D > 0$

$D < 0$
Frequency response function

\[ |G(i\omega)| \]

\( \beta = 1 \)
\( \beta = 0.75 \)
\( \beta = 0.5 \)
\( \beta = 0.25 \)
\( \beta = 0 \) (viscous)

(a) \( \zeta = 0.1 \) (b) \( \zeta = 0.25 \) (c) \( \zeta = 0.5 \) (d) \( \zeta = 1.0 \)
Partial summary of new results

- A non-viscously damped oscillator will have oscillatory motion if $\zeta < \frac{4}{3\sqrt{3}}$ or $\beta > \frac{1}{3\sqrt{3}}$.

- If $\beta < \frac{1}{3\sqrt{3}}$, the oscillatory motion is possible if and only if $\zeta \notin [\zeta_L, \zeta_U]$. $\zeta_L$ and $\zeta_U$ are the lower and upper critical damping factors.

- If $\beta > 1/4$, the natural frequency of a non-viscously damped oscillator will be more than that of an equivalent undamped oscillator.

- The amplitude of the frequency response function of a non-viscously damped oscillator can reach a maximum value if $\zeta < \frac{1}{2} \sqrt{\sqrt{5} - 1}$ or $\beta > \frac{1}{2} \sqrt{3\sqrt{3} - 4}$.
Some References

Equation of motion

The governing equation is

\[ m \frac{d^2 x}{dt^2} + c \int_{\tau=0}^{\tau=t} \mu e^{-\mu (\tau - \dot{\tau})} \frac{dx}{d\dot{\tau}} d\dot{\tau} + \alpha_1 k x + \alpha_2 k x^3 = A \cos(\Omega \dot{t}), \]

- \( x \): the displacement of mass \( m \); \( k \): linear spring stiffness
- \( \alpha_1, \alpha_2 \): strength of linear and nonlinear spring stiffness
- \( c \): viscous damping coefficient
- The non-viscous damping effect is represented by the parameter \( \mu \) via the convolution integral. \( \mu \rightarrow \infty \) implies viscous damping, i.e., classical Duffing oscillator
The nondimensional equation

The nondimensional governing equation is

$$\ddot{x} + 2\zeta \int_{0}^{t} e^{-\frac{1}{\beta}(t-\tau)} \frac{\dot{x}d\tau}{\beta} + \alpha_1 x + \alpha_2 x^3 = x_0 \cos(\omega t),$$

We now define the integral term as

$$y = \int_{0}^{t} e^{-\frac{1}{\beta}(t-\tau)} \frac{\dot{x}d\tau}{\beta}$$

Then by using the Leibniz rule for differentiation of an integral we can write

$$\dot{y} = \frac{1}{\beta} \dot{x} - \frac{1}{\beta} x$$
The first-order form

We can then write a set of three first order ordinary differential equations

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -2\zeta y - \alpha_1 x_1 - \alpha_2 x_1^3 + x_0 \cos(\omega t), \\
\dot{y} &= \frac{1}{\beta} x_2 - \frac{1}{\beta} y,
\end{align*}
\]

Note that if we multiply through the last line by \(\beta\), then as \(\beta \to 0\), \(y \to x_2\) and the viscous damping case is obtained.
Computing solutions

- 4th order Runge-Kutta integration algorithm
- Start at the lowest $\omega$ value
- Compute transient periods (typically 100–200)
- Max displacement recorded for 20–50 steady state periods
- Increase $\omega$ and repeat
- At max $\omega$, reverse increment and the process continued to $\omega_{min}$
Weak coupling: $\alpha_1 = 1.0$ and $\alpha_2 = 0.05$
Strong coupling: $\alpha_1 = 0$ and $\alpha_2 = 1$

(a) Viscous case: $\beta = 0$
(b) $\beta = 0.1$
(c) $\beta = 0.3$
(d) $\beta = 0.8$

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Conclusions

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- Many new features cannot be predicted (or institutively guessed) by 'simple extension' of the classical results known for viscously damped systems.
- More new dynamical features are yet to be discovered in the future ... this is far from over!