Response Variability of Viscoelastically Damped Systems

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Outline of the presentation

- Overview of viscoelastically damped systems
- Eigensolutions
  - State-space approach
  - Approximate methods in N-space
- Dynamic response calculation
- Parametric sensitivity of eigensolutions
- Parametric sensitivity of dynamic response
- Numerical results
- Conclusions
Damping models

- Viscous damping is the most widely used damping model for complex aerospace dynamic systems.
- In general a physically realistic model of damping may not be a viscous damping model.
- Damping models in which the dissipative forces depend on any quantity other than the instantaneous generalized velocities are non-viscous (e.g., viscoelastic) damping models.
- Possibly the most general way to model damping within the linear range is to use non-viscous damping models which depend on the past history of motion via convolution integrals over kernel functions.
Equation of motion

The equations of motion of a $N$-DOF linear system:

$$M \ddot{u}(t) + \int_0^t G(t-\tau) \dot{u}(\tau) \, d\tau + Ku(t) = f(t)$$

(1)

together with the initial conditions

$$u(t=0) = u_0 \in \mathbb{R}^N \quad \text{and} \quad \dot{u}(t=0) = \dot{u}_0 \in \mathbb{R}^N.$$  

(2)

$u(t)$: displacement vector, $f(t)$: forcing vector, $M, K$: mass and stiffness matrices.

In the limit when $G(t-\tau) = C \delta(t-\tau)$, where $\delta(t)$ is the Dirac-delta function, this reduces to viscous damping.
## Damping functions - 1

<table>
<thead>
<tr>
<th>Model Number</th>
<th>Damping function</th>
<th>Author and year of publication</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$G(s) = \sum_{k=1}^{n} \frac{a_k s}{s + b_k}$</td>
<td>Biot[1] - 1955</td>
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<tr>
<td>2</td>
<td>$G(s) = \frac{E_1 s^\alpha - E_0 b s^\beta}{1 + b s^\beta}$ (0 &lt; $\alpha$, $\beta$ &lt; 1)</td>
<td>Bagley and Torvik[2] - 1983</td>
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<td>3</td>
<td>$sG(s) = G^\infty \left[ 1 + \sum_{k} \alpha_k \frac{s^2 + 2\xi_k \omega_k s}{s^2 + 2\xi_k \omega_k s + \omega_k^2} \right]$</td>
<td>Golla and Hughes[3] - 1985 and McTavish and Hughes[4] - 1993</td>
</tr>
<tr>
<td>4</td>
<td>$G(s) = 1 + \sum_{k=1}^{n} \frac{\Delta_k s}{s + \beta_k}$</td>
<td>Lesieutre and Mingori[5] - 1990</td>
</tr>
<tr>
<td>5</td>
<td>$G(s) = c \frac{1 - e^{-st_0}}{st_0}$</td>
<td>Adhikari[6] - 1998</td>
</tr>
<tr>
<td>6</td>
<td>$G(s) = \frac{c}{st_0} \frac{1 + 2(st_0/\pi)^2 - e^{-st_0}}{1 + 2(st_0/\pi)^2}$</td>
<td>Adhikari[6] - 1998</td>
</tr>
<tr>
<td>7</td>
<td>$G(s) = c e^{s^2/4\mu} \left[ 1 - \text{erf} \left( \frac{s}{2\sqrt{\mu}} \right) \right]$</td>
<td>Adhikari and Woodhouse[7] - 2001</td>
</tr>
</tbody>
</table>

Some damping functions in the Laplace domain.
We use a damping model for which the kernel function matrix:

\[
G(t) = \sum_{k=1}^{n} \mu_k e^{-\mu_k t} C_k
\]  

(3)

The constants \( \mu_k \in \mathbb{R}^+ \) are known as the relaxation parameters and \( n \) denotes the number relaxation parameters.

When \( \mu_k \to \infty, \forall k \) this reduces to the viscous damping model:

\[
C = \sum_{k=1}^{n} C_k.
\]  

(4)
Non-linear Eigenvalue Problem

The eigenvalue problem associated with a linear system with exponential damping model:

\[
\begin{bmatrix}
s_j^2M + s_j \sum_{k=1}^{n} \frac{\mu_k}{s_j + \mu_k} C_k + K
\end{bmatrix} z_j = 0, \quad \text{for } j = 1, \cdots, m.
\]

Two types of eigensolutions:

- \(2N\) complex conjugate solutions - underdamped/vibrating modes
- \(p\) real solutions \([p = \sum_{k=1}^{n} \text{rank } (C_k)]\) - overdamped modes
State-space Approach - 1

The equation of motion can be transformed to \((m = 2N + nN)\) dimensional system

\[
B \ddot{z}(t) = A z(t) + r(t)
\]  \(\quad (6)\)

\[
B = \begin{bmatrix}
\sum_{k=1}^{n} C_k & M & -C_1/\mu_1 & \cdots & -C_n/\mu_n \\
M & O & O & O & O \\
-C_1/\mu_1 & O & C_1/\mu_1^2 & O & O \\
\vdots & O & O & \ddots & O \\
-C_n/\mu_n & O & O & O & C_n/\mu_n^2
\end{bmatrix}, \quad r(t) = \begin{bmatrix} f(t) \\
0 \\
0 \\
\vdots \\
0 \end{bmatrix}
\]  \(\quad (7)\)

\[
A = \begin{bmatrix}
-K & O & O & O & O \\
0 & M & O & O & O \\
O & O & -C_1/\mu_1 & O & O \\
O & O & O & \ddots & O \\
O & O & O & O & -C_n/\mu_n
\end{bmatrix}, \quad z(t) = \begin{bmatrix} u(t) \\
v(t) \\
y_1(t) \\
\vdots \\
y_n(t) \end{bmatrix}
\]  \(\quad (8)\)
The eigenvalue problem in the state-space is given by

\[ A z_j = \lambda_j B z_j \]  \hspace{1cm} (9)

The ‘size’ of the eigenvalue problem is \((2N + nN)\)-dimensional.

- although exact in nature, the state-space approach is computationally very intensive for real-life systems;
- the physical insights offered by methods in the original space (eg, the modal analysis) is lost in a state-space based approach
Approximate eigensolutions

If $\omega_j$ and $x_j$ are the undamped natural frequency and mode shape of the system satisfying $Kx_j = \omega_j^2 Mx_j$, the eigenvalues of the viscoelastically damped system obtained using the first-order perturbation method:

$$s_j \approx i\omega_j - G'_{jj}(i\omega_j)/2, \quad -i\omega_j - G'_{jj}(-i\omega_j)/2. \quad (10)$$

Similarly, the eigenvectors are given by

$$z_j \approx x_j - \sum_{k=1}^{N} \sum_{k \neq j} \frac{s_j G'_{kj}(s_j)x_k}{\omega_k^2 + s_j^2 + s_j G'_{kk}(s_j)}. \quad (11)$$
Taking the Laplace transform of the equation of motion and considering the initial conditions we have

\[ s^2 M \ddot{q} - s M q_0 - M \dot{q}_0 + s G(s) \ddot{q} - G(s) q_0 + K \ddot{q} = \bar{f}(s) \]

or

\[ D(s) \ddot{q} = \bar{f}(s) + M \dot{q}_0 + [sM + G(s)] q_0. \]

The *dynamic stiffness matrix* is defined as

\[ D(s) = s^2 M + s G(s) + K \in \mathbb{C}^{N \times N}. \tag{12} \]

The inverse of the dynamics stiffness matrix, known as the transfer function matrix, is given by

\[ H(s) = D^{-1}(s) \in \mathbb{C}^{N \times N}. \tag{13} \]
Using the residue-calculus, the transfer function matrix can be expressed like a viscously damped system as

\[
H(s) = \sum_{j=1}^{m} \frac{R_j}{s - s_j}; \quad R_j = \text{res}_{s=s_j} [H(s)] = \frac{z_j z_j^T}{z_j^T \frac{\partial D(s_j)}{\partial s_j} z_j}
\]  

(14)

where \( m \) is the number of non-zero eigenvalues (order) of the system, \( s_j \) and \( z_j \) are respectively the eigenvalues and eigenvectors of the system, which are solutions of the non-linear eigenvalue problem

\[
[s_j^2 \mathbf{M} + s_j \mathbf{G}(s_j) + \mathbf{K}] \mathbf{z}_j = 0, \quad \text{for } j = 1, \cdots, m
\]

(15)
The expression of $H(s)$ allows the response to be expressed as modal summation as

$$\overline{q}(s) = \sum_{j=1}^{m} \gamma_j \frac{z_j^T \bar{f}(s) + z_j^T M \dot{q}_0 + s z_j^T M q_0 + z_j^T G(s) q_0(s)}{s - s_j} z_j$$  \hspace{1cm} (16)$$

where the normalization constant

$$\gamma_j = \frac{1}{z_j^T \frac{\partial D(s_j)}{\partial s_j} z_j}.$$  \hspace{1cm} (17)$$

We use the approximate eigensolutions in the ‘N’-space.
The response in the time domain can be obtained by taking the inverse transform:

\[ q(t) = \mathcal{L}^{-1}[\tilde{q}(s)] = \sum_{j=1}^{m} \gamma_j a_j(t)z_j \]  

(18)

where the time-dependent scalar coefficients (for \( t > 0 \))

\[ a_j(t) = \int_{0}^{t} e^{s_j(t-\tau)} \left\{ z_j^T f(\tau) + z_j^T G(\tau)q_0 \right\} d\tau + e^{s_j t} \left\{ z_j^T M\dot{q}_0 + s_j z_j^T M\dot{q}_0 \right\} . \]  

(19)
Response variability: Direct approach

The dynamic response in the Laplace domain:

$$\bar{q}(s) = D^{-1}(s)\bar{p}(s) \quad (20)$$

where

$$D(s) = s^2M + s \sum_{k=1}^{n} \frac{\mu_k}{s + \mu_k} C_k + K \quad (21)$$

$$\bar{p}(s) = \bar{f}(s) + M \dot{q}_0 + [sM + G(s)] q_0. \quad (22)$$

Suppose the system matrices are functions of some design parameter $p$. We want to obtain $\frac{\partial \bar{q}(s)}{\partial p}$. 
Response variability: Direct approach

Differentiating the equation of motion in the Laplace domain

\[
\frac{\partial \bar{q}(s)}{\partial p} = \frac{\partial D^{-1}(s)}{\partial p} \bar{p}(s) + D^{-1}(s) \frac{\partial \bar{p}(s)}{\partial p}
\]  

(23)

Using the direct approach,

\[
\frac{\partial D^{-1}(s)}{\partial p} = D^{-1}(s) \frac{\partial D(s)}{\partial p} D^{-1}(s)
\]  

(24)

where

\[
\frac{\partial D(s)}{\partial p} = s^2 \frac{\partial M}{\partial p} + s \frac{\partial}{\partial p} \left\{ \sum_{k=1}^{n} \frac{\mu_k}{s + \mu_k} C_k \right\} + \frac{\partial K}{\partial p}
\]  

(25)
Response variability: Modal approach

\[
D^{-1}(s) = \sum_{j=1}^{m} \frac{R_j}{s - s_j}; \quad R_j = \frac{z_j z_j^T}{\theta_j}
\]  \hspace{1cm} (26)

Using the modal approach,

\[
\frac{\partial D^{-1}(s)}{\partial p} = \sum_{j=1}^{m} \frac{\partial R_j}{\partial p} \frac{1}{s - s_j} - \frac{R_j}{(s - s_j)^2} \frac{\partial s_j}{\partial p}
\]  \hspace{1cm} (27)

\[
\frac{\partial R_j}{\partial p} = \left( \frac{\partial z_j}{\partial p} z_j^T + z_j \frac{\partial z_j^T}{\partial p} \right) / \theta_j
\]  \hspace{1cm} (28)
Eigensolution derivative

It can be shown that (Adhikari: AIAA Journal, 40[10] (2002), pp. 2061-2069)

\[
\frac{\partial s_j}{\partial p} = -\frac{1}{\theta_j} \left( z_j^T \frac{\partial D(s)}{\partial p} \bigg|_{s=s_j} z_j \right).
\]  (29)

\[
\frac{\partial z_j}{\partial p} = a_{jj} z_j - \sum_{\substack{k=1 \atop k \neq j}}^{m} \frac{u_k^T \frac{\partial D(s)}{\partial p} \bigg|_{s=s_j} z_j}{\theta_k (s_j - \lambda_k)} u_k
\]  (30)

where

\[
a_{jj} = -\frac{z_j^T \frac{\partial^2 [D(s)]}{\partial s \partial p} \bigg|_{s=s_j} z_j}{2 \left( z_j^T \frac{\partial D(s)}{\partial s} \bigg|_{s=s_j} z_j \right)}.
\]  (31)
Example of a 2 DOF system

The two degrees-of-freedom spring-mass system with non-viscous damping, $m = 1$ Kg, $k_1 = 1000$ N/m, $k_3 = 100$ N/m, $g(t) = c \left( \mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t} \right)$, $c = 4.0$ Ns/m, $\mu_1 = 10.0$ s$^{-1}$, $\mu_2 = 2.0$ s$^{-1}$
Example: system matrices

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_3 & -k_3 \\ -k_3 & k_2 + k_3 \end{bmatrix} \quad (32)$$

and

$$\mathcal{G}(t) = g(t)\hat{I}, \quad \text{where} \quad \hat{I} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (33)$$

The damping function $g(t)$ is assumed to be the GHM model[3, 4] so that

$$g(t) = c \left( \mu_1 e^{-\mu_1 t} + \mu_2 e^{-\mu_2 t} \right); \quad c, \mu_1, \mu_2 \geq 0, \quad (34)$$
System matrix derivative

We consider the derivative of eigenvalues with respect to the relaxation parameter $\mu_1$. The derivative of the system matrices:

$$\frac{\partial M}{\partial \mu_1} = 0, \quad \frac{\partial G(s)}{\partial \mu_1} = \hat{I} \frac{c \, s}{(s + \mu_1)^2} \quad \text{and} \quad \frac{\partial K}{\partial \mu_1} = 0.$$  \hspace{1cm} (35)

Thus we have

$$\frac{\partial G(s)}{\partial s} = -\hat{I}c \left\{ \frac{\mu_1}{(s + \mu_1)^2} + \frac{\mu_2}{(s + \mu_2)^2} \right\}$$

$$\frac{\partial^2 [G(s)]}{\partial s \partial \mu_1} = -\hat{I}c \frac{s - \mu_1}{(s + \mu_1)^3}. \hspace{1cm} (36)$$
Real part of the derivative of the first eigenvalue with respect to the relaxation parameter $\mu_1$. 
Real part of the derivative of the second eigenvalue with respect to the relaxation parameter $\mu_1$. 
Numerical Results

Imaginary part of the derivative of the first eigenvalue with respect to the damping parameters $c$, $\mu_1$ and $\mu_2$. 
Numerical Results

Imaginary part of the derivative of the second eigenvalue with respect to the damping parameters $c$, $\mu_1$ and $\mu_2$. 
Numerical Results

Real part of the derivative of the first eigenvector with respect to $k_2$. 
Numerical Results

Derivative of the second eigenvector

\[ \frac{dU_{12}}{dk_2}, \quad \frac{dU_{22}}{dk_2} \]

viscously damped \[ \frac{dU_{12}}{dk_2}, \quad \frac{dU_{22}}{dk_2} \]

Real part of the derivative of the second eigenvalue with respect to \( k_2 \).
Conclusions - 1

- Multiple degree-of-freedom linear systems with viscoelastic damping is considered.
- The transfer function matrix of the system was derived in terms of the eigenvalues and eigenvectors of the second-order system.
- The eigensolutions are obtained using an approximate perturbation method (although an exact but computationally more expensive state-space method can be used).
Parametric sensitivity of the dynamic response was derived using two approaches - namely the direct approach and modal approach.

The direct approach is easy to implement but computationally expensive as one has to differentiate the dynamic stiffness matrix at every frequency point.

The modal approach utilizes derivatives of the complex eigensolutions and generally computationally more efficient.
Future Directions

- The results derived here extend the equivalent results available for viscously damped systems. The expressions can be useful to any problems which require parametric sensitivity information. Such problems include (a) probabilistic analysis, (b) optimal design, (c) model updating and system identification.

- Future work will look into (a) sensitivity of transient dynamic response of viscoelastically damped systems in the time domain (this problem has relevance to vehicle noise reduction), and (b) joint sensitivity analysis of multiple parameters.
References


