Random Eigenvalue Problems in Structural Dynamics: An Experimental Investigation

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Outline of the presentation

- A Brief Overview of Random Eigenvalue Problems
- Random Eigenvalues of a Fixed-Fixed Beam
- Random Eigenvalues of a cantilever plate
- System Model and Experimental Setup
- Experimental methodology
- Eigenvalue Statistics
  - Experimental results
  - Monte Carlo simulation
- Conclusions & future directions
Ensembles of structural dynamical systems

Many structural dynamic systems are manufactured in a production line (nominally identical systems)
A complex structural dynamical system

Complex aerospace system can have millions of degrees of freedom and significant ‘errors’ and/or ‘lack of knowledge’ in its numerical (Finite Element) model.
Sources of uncertainty

(a) **parametric uncertainty** - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
(b) **model inadequacy** - arising from the lack of scientific knowledge about the model which is a-priori unknown;
(c) **experimental error** - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
(d) **computational uncertainty** - e.g., machine precession, error tolerance and the so-called ‘h’ and ‘p’ refinements in finite element analysis, and
(e) **model uncertainty** - genuine randomness in the model such as uncertainty in the position and velocity in quantum mechanics, deterministic chaos.
Overview of Random Eigenvalue Problems

- EVP of Undamped or proportionally damped systems:

\[ K\phi_j = \lambda_j M\phi_j \]  \hspace{1cm} (1)

- \( \lambda_j \): Eigenvalue (natural frequency squared)
- \( \phi_j \): Eigenvector (modeshape)
- \( M \& K \) are symmetric and P.D random matrices \( \Rightarrow \) \( \lambda_j \) real and positive.

\[ M = \overline{M} + \delta M \quad \text{and} \quad K = \overline{K} + \delta K. \] \hspace{1cm} (2)

\( \overline{(*)} \): Nominal (deterministic) of of \( (*) \)
\( \delta(*) \): Random parts of \( (*) \).
Randomness

\[ M = \bar{M} + \delta M \quad \text{and} \quad K = \bar{K} + \delta K. \]

- \( \delta M \) and \( \delta K \) are zero-mean random matrices.
- Small randomness assumption that preserve symmetry and P.D of \( M \) and \( \bar{M} \).
- No assumptions on the type of randomness: need not be Gaussian, for example
- Fixed-Fixed beam with random placement of equal masses gives \( \delta M \neq 0 \delta K = 0 \)
- Cantilever plate with random placement of random oscillators gives \( \delta M \neq 0 \delta K \neq 0 \)
Fixed-Fixed Beam: Experiments

The test rig for the fixed-fixed beam
Actuator: Shaker, Sensors: Accelerometers
Fixed-Fixed Beam: Experiments

Attached masses (magnets) at random locations.
12 masses, each weighting 2g, are used.
## Fixed-Fixed Beam: Properties

<table>
<thead>
<tr>
<th>Beam Properties</th>
<th>Numerical values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ($L$)</td>
<td>1200 mm</td>
</tr>
<tr>
<td>Width ($b$)</td>
<td>40.06 mm</td>
</tr>
<tr>
<td>Thickness ($t_h$)</td>
<td>2.05 mm</td>
</tr>
<tr>
<td>Mass density ($\rho$)</td>
<td>7800 Kg/m$^3$</td>
</tr>
<tr>
<td>Young’s modulus ($E$)</td>
<td>$2.0 \times 10^5$ MPa</td>
</tr>
<tr>
<td>Total weight</td>
<td>0.7687 Kg</td>
</tr>
</tbody>
</table>

Material and geometric properties of the beam.
Shaker as an Impulse Hammer

pulse rate: 20s & pulse width: 0.01s. Eliminate input uncertainties.

brass plate (2g) takes impact.
Experiments: Protocol

- Arrange the masses along the beam at random locations (computer generated)
- Measure impulse response at: 23 cm (Point1) 50 cm (Point2, also the actuation point) and 102 cm (Point3) from the left end of the beam in a 32 channel LMS™ system
- Transform to frequency domain to estimate frequency response function (FRF).
- Curvefit the FRF to estimate the natural frequencies $\omega_n$ and damping factors $Q_n$
  - Rational Fraction Polynomial (RFP) method
  - Nonlinear Leastsquares method
- Calculate the statistics of natural frequencies
Experiments: FRF at Point 1 (23 cm from the left end)
Experiments: FRF at point 2 (the driving point FRF, 50 cm from the left end)

Log amplitude (dB) of $H_{(22)}(\omega)$

Frequency (Hz)

Palm Springs, CA, 6th May 2009
Experiments: FRF at point 3 (102 cm from the left end)
Ensemble Mean

Left: RFP; Right: Nonlinear Least squares
Standard Deviation

Left: RFP; Right: Nonlinear Least-squares
PDFs

Left: RFP; Right: Nonlinear Least-squares
Cantilever plate
# Cantilever plate: Properties

<table>
<thead>
<tr>
<th>Plate Properties</th>
<th>Numerical values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length ($L$)</td>
<td>998 mm</td>
</tr>
<tr>
<td>Width ($b$)</td>
<td>530 mm</td>
</tr>
<tr>
<td>Thickness ($t_h$)</td>
<td>3.0 mm</td>
</tr>
<tr>
<td>Mass density ($\rho$)</td>
<td>7800 Kg/m$^3$</td>
</tr>
<tr>
<td>Young’s modulus ($E$)</td>
<td>$2.0 \times 10^5$ MPa</td>
</tr>
<tr>
<td>Total weight</td>
<td>12.38 Kg</td>
</tr>
</tbody>
</table>

Table 1: Material and geometric properties of the cantilever plate considered for the experiment
Attached oscillators at random locations. The spring stiffness varies so...
## Properties of Attached Oscillators

<table>
<thead>
<tr>
<th>Oscillator Number</th>
<th>Spring stiffness (×10⁴ N/m)</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.6800</td>
<td>59.2060</td>
</tr>
<tr>
<td>2</td>
<td>0.9100</td>
<td>43.5744</td>
</tr>
<tr>
<td>3</td>
<td>1.7030</td>
<td>59.6099</td>
</tr>
<tr>
<td>4</td>
<td>2.4000</td>
<td>70.7647</td>
</tr>
<tr>
<td>5</td>
<td>1.5670</td>
<td>57.1801</td>
</tr>
<tr>
<td>6</td>
<td>2.2880</td>
<td>69.0938</td>
</tr>
<tr>
<td>7</td>
<td>1.7030</td>
<td>59.6099</td>
</tr>
<tr>
<td>8</td>
<td>2.2880</td>
<td>69.0938</td>
</tr>
<tr>
<td>9</td>
<td>2.1360</td>
<td>66.7592</td>
</tr>
<tr>
<td>10</td>
<td>1.9800</td>
<td>64.2752</td>
</tr>
</tbody>
</table>

**Table 2:** Stiffness of the springs and natural frequency of the oscillators used to simulate unmodelled dynamics (the mass of each oscillator is 121.4g).
Experiments: Protocol

- Attach the oscillators at random locations (computer generated)
- Measure impulse response at: Point 1: (4,6), Point 2: (6,11), Point 3: (11,3), Point 4: (14,14), Point 5: (18,2), Point 6: (21,10)
- Transform to frequency domain to estimate frequency response function (FRF).
- Curvefit the FRF to estimate the natural frequencies $\omega_n$ and damping factors $Q_n$
  - Rational Fraction Polynomial (RFP) method
  - Nonlinear Least squares method
- Calculate the statistics of natural frequencies
Experiments: FRF at Point 1
Experiments: FRF at point 3

Palm Springs, CA, 6th May 2009
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Ensemble Mean

Left: RFP; Right: Nonlinear Leastsquares
Standard Deviation

Left: RFP; Right: Nonlinear Least-squares
PDFs

Left: RFP; Right: Nonlinear Least-squares
Conclusions

- The ensemble statistics such as mean and standard deviation for natural frequencies vary with the spatial location of the measured FRFs and the type of the system identification technique chosen to estimate the natural frequencies.

- Whilst a reasonable predictions for the mean and the standard deviations may be obtained using the Monte Carlo Simulation, higher moments, and hence the pdfs can be significantly different.

- In some cases, the differences in pdfs arising from different points and different identification methods can be more than those obtained from the Monte Carlo Simulation.