An Efficient Computational Solution Scheme of the Random Eigenvalue Problems

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Outline

- Introduction
- Random Eigenvalue Problem
- High Dimensional Model Representation (HDMR)
- Examples
- Conclusions
Sources of uncertainty

- (a) **parametric uncertainty** - e.g., uncertainty in geometric parameters, friction coefficient, strength of the materials involved;
- (b) **model inadequacy** - arising from the lack of scientific knowledge about the model which is a-priori unknown;
- (c) **experimental error** - uncertain and unknown error percolate into the model when they are calibrated against experimental results;
- (d) **computational uncertainty** - e.g., machine precession, error tolerance and the so called ‘h’ and ‘p’ refinements in finite element analysis,
Random Eigenvalue Problem

\[ M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = p(t) \]

- Due to the presence of uncertainties, mass, damping and stiffness matrices are random matrices.
- The primary objectives are
  - To quantify the uncertainties in system matrices.
  - To estimate the variability of system responses.
Random Eigenvalue Problem

- **Random eigenvalue of linear structural system**

\[ K(X)\Phi(X) = \Lambda(X)M(X)\Phi(X) \]

- **Main issues**
  - To find probabilistic characteristics of eigenpair.
  - To find the joint statistics (moments, correlation).
  - Several approaches are available on random eigenvalue problem, which are based on
    - Perturbation method (Boyce, 1968; Zhang & Ellingwood, 1995)
    - Iteration method (Boyce, 1968)
    - Ritz method (Mehlhose, 1999)
    - Crossing theory (Grigorie, 1992)
    - Stochastic reduced basis (Nair & Keane, 2003)
    - Asymptotic method (Adhikari, 2006)
Perturbation Method

Taylor series expansion of $\lambda_j(x)$ about $x = \alpha$

$$
\lambda_j(x) \approx \lambda_j(\alpha) + d^T_{\lambda_j}(\alpha)(x - \alpha) + \frac{1}{2}(x - \alpha)^T D_{\lambda_j}(\alpha)(x - \alpha)
$$

In the mean-centered approach $\alpha$ is the mean of $x$

Alternatively, $\alpha$ can be obtained such that the any moment of each eigenvalue is calculated most accurately
We want to evaluate an $m$-dimensional integral over the unbounded domain $\mathbb{R}^m$:

$$ \mathcal{J} = \int_{\mathbb{R}^m} e^{-f(x)} \, dx $$

- Assume $f(x)$ is smooth and at least twice differentiable.
- The maximum contribution to this integral comes from the neighborhood where $f(x)$ reaches its global minimum, say $\theta \in \mathbb{R}^m$. 
Multidimensional Integrals

Therefore, at \( x = \theta \)

\[
\frac{\partial f(x)}{\partial x_k} = 0, \forall k \quad \text{or} \quad d_f(\theta) = 0
\]

Expand \( f(x) \) in a Taylor series about \( \theta \):

\[
\mathcal{J} = \int_{\mathbb{R}^m} e^{-\{f(\theta) + \frac{1}{2}(x-\theta)^T D_f(\theta)(x-\theta) + \varepsilon(x,\theta)\}} \, dx
\]

\[
= e^{-f(\theta)} \int_{\mathbb{R}^m} e^{-\frac{1}{2}(x-\theta)^T D_f(\theta)(x-\theta) - \varepsilon(x,\theta)} \, dx
\]
Multidimensional Integrals

- Use the coordinate transformation:
  \[ \xi = (x - \theta) D_f^{-1/2}(\theta) \]

- The Jacobian: \[ ||J|| = ||D_f(\theta)||^{-1/2} \]

- The integral becomes:
  \[ J \approx e^{-f(\theta)} \int_{\mathbb{R}^m} ||D_f(\theta)||^{-1/2} e^{-\frac{1}{2} (\xi^T \xi)} d\xi \]

or

\[ J \approx (2\pi)^{m/2} e^{-f(\theta)} ||D_f(\theta)||^{-1/2} \]
Moments of Eigenvalues

An arbitrary $r$th order moment of the eigenvalues can be obtained from

$$
\mu_j^{(r)} = E \left[ \lambda_j^r(x) \right] = \int_{\mathbb{R}^m} \lambda_j^r(x) p_x(x) \, dx
$$

$$
= \int_{\mathbb{R}^m} e^{-\left( L(x) - r \ln \lambda_j(x) \right)} \, dx, \quad r = 1, 2, 3 \cdots
$$

Previous result can be used by choosing

$$
f(x) = L(x) - r \ln \lambda_j(x)
$$
Moments of Eigenvalues

After some simplifications

\[
\mu_j^{(r)} \approx (2\pi)^{m/2} \lambda_j^r(\theta)e^{-L(\theta)}
\]

\[
\left\| D_L(\theta) + \frac{1}{r}d_L(\theta)d_L(\theta)^T - \frac{r}{\lambda_j(\theta)}D\lambda_j(\theta) \right\|^{-1/2}
\]

\[ r = 1, 2, 3, \ldots \]

\[ \theta \text{ is obtained from:} \]

\[ d\lambda_j(\theta)_r = \lambda_j(\theta)d_L(\theta) \]
Moments of Eigenvalues

- Mean of the eigenvalues:

\[ \hat{\lambda}_j = \mu_j^{(1)} = \lambda_j(\theta)e^{-L(\theta)} \]

\[ \left\| D_L(\theta) + d_L(\theta)d_L(\theta)^T - D_{\lambda_j}(\theta)/\lambda_j(\theta) \right\|^{-1/2} \]

- Central moments of the eigenvalues:

\[ E \left[ \left( \lambda_j - \hat{\lambda}_j \right)^r \right] = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mu_j^{(k)} \hat{\lambda}_j^{r-k} \]
Multivariate Gaussian Case

\[ L(x) = \frac{m}{2} \ln(2\pi) + \frac{1}{2} \ln \|\Sigma\| + \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \]

so \( d_L(x) = \Sigma^{-1} x \) and \( D_L(x) = \Sigma^{-1} \). Therefore:

\[ \mu_j^{(r)} \approx \lambda_j^r(\theta) e^{-\frac{1}{2}(\theta - \mu)^T \Sigma^{-1} (\theta - \mu)} \]

\[ \left\| I + \frac{1}{r} \theta \theta^T \Sigma^{-1} - \frac{r}{\lambda_j(\theta)} \Sigma D \lambda_j(\theta) \right\|^{-1/2} \]

where \( \theta = \frac{r}{\lambda_j(\theta)} \Sigma d \lambda_j(\theta) \)
Constraints for $u \in [0, \infty]$:

$$\int_0^\infty p_{\lambda_j}(u) du = 1$$

$$\int_0^\infty u^r p_{\lambda_j}(u) du = \mu_j^{(r)}, \quad r = 1, 2, 3, \cdots, n$$

Maximizing Shannon’s measure of entropy $S = - \int_0^\infty p_{\lambda_j}(u) \ln p_{\lambda_j}(u) du$, the pdf of $\lambda_j$ is

$$p_{\lambda_j}(u) = e^{-\{\rho_0 + \sum_{i=1}^n \rho_i u^i\}} = e^{-\rho_0} e^{-\sum_{i=1}^n \rho_i u^i}, \quad u \geq 0$$
Taking first two moments, the resulting pdf is a truncated Gaussian density function

\[
p\lambda_j(u) = \frac{1}{\sqrt{2\pi}\sigma_j \Phi \left( \frac{\lambda_j}{\sigma_j} \right)} \exp \left\{ -\frac{\left( u - \lambda_j \right)^2}{2\sigma_j^2} \right\}
\]

where \( \sigma_j^2 = \mu_j^{(2)} - \lambda_j^2 \)

- Ensures that the probability of any eigenvalues becoming negative is zero
Maximum Entropy pdf

- With three moments

Pdf of $j$th eigenvalue

$$p_{\lambda_j}(u) \approx \frac{1}{\gamma_j} p_{\chi_j^2} \left( \frac{u - \eta_j}{\gamma_j} \right) = \frac{(u - \eta_j)^{\nu_j/2 - 1} e^{-(u - \eta_j)/2\gamma_j}}{(2\gamma_j)^\nu_j/2 \Gamma(\nu_j/2)}$$

The constants $\eta_j$, $\gamma_j$, and $\nu_j$ are such that the first three moments of $\lambda_j$ are the same.
**HDMR**

Input $\mathbf{x} \in \mathbb{R}^N$ → **SYSTEM** → Output $y(\mathbf{x}) \in \mathbb{R}$

$y(\mathbf{x}) = y_0 + \sum_{i=1}^{N} y_i(x_i) + \sum_{i_1,i_2=1}^{N} y_{i_1i_2}(x_{i_1}, x_{i_2}) + \cdots + \sum_{i_1<i_2<\cdots<i_S=1}^{N} y_{i_1\cdots i_S}(x_{i_1},\cdots,x_{i_S}) + \cdots + y_{12\cdots N}(x_1,\cdots,x_N)$

First-order

$= \hat{y}_1(\mathbf{x})$

$= \hat{y}_2(\mathbf{x})$

$= \hat{y}_3(\mathbf{x})$

Second-order (2D cooperative effects)

$= \hat{y}_2(\mathbf{x})$

$= \hat{y}_3(\mathbf{x})$

**S-order**

(SD cooperative effects)

**Conjecture:** Component functions arising in proposed decomposition will exhibit insignificant $S$-order effects cooperatively when $S \rightarrow N$.

(Rabitz & Alis, 1999; Alis & Rabitz, 2001)
HDMR

- Lower-order Approximations

First-order Approximation

\[
\hat{y}^I(x) = \hat{y}^I(x_1, \ldots, x_N) = \sum_{i=1}^{N} y(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N) - (N-1)y(c) = y_i(x_i)
\]

Second-order Approximation

\[
\hat{y}^{II}(x) = \hat{y}^{II}(x_1, \ldots, x_N) = \sum_{i_1, i_2=1 \atop i_1 < i_2}^{N} y(c_1, \ldots, c_{i-1}, x_{i_1}, c_{i_1+1}, \ldots, c_{i_2-1}, x_{i_2}, c_{i_2+1}, \ldots, c_N) + (N-2)\sum_{i=1}^{N} y(c_1, \ldots, c_{i-1}, x_i, c_{i+1}, \ldots, c_N) + \frac{(N-1)(N-2)}{2} y(c) = y_i(x_i)
\]

Interpolation function

\[
y_i(x_i) \approx \sum_{j=1}^{n} \phi_j(x_i) y(c_1, \ldots, c_{i-1}, x_i^{(j)}, c_{i+1}, \ldots, c_N)
\]

\[
y_{i_1i_2}(x_{i_1}, x_{i_2}) \approx \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \phi_{j_1j_2}(x_{i_1}, x_{i_2}) y(c_1, \ldots, c_{i_1-1}, x_{i_1}^{(j_1)}, c_{i_1+1}, \ldots, c_{i_2-1}, x_{i_2}^{(j_2)}, c_{i_2+1}, \ldots, c_N)
\]
Convergence Issue

- Two-dimensional Taylor Series Expansion

\[ y(x_1, x_2) = y(c_1, c_2) + \frac{\partial y(c_1, c_2)}{\partial x_1}(x_1 - c_1) + \frac{\partial y(c_1, c_2)}{\partial x_2}(x_2 - c_2) + \frac{\partial^2 y(c_1, c_2)}{\partial x_1^2}(x_1 - c_1)^2 + \frac{\partial^2 y(c_1, c_2)}{\partial x_2^2}(x_2 - c_2)^2 + \frac{\partial^2 y(c_1, c_2)}{\partial x_1 \partial x_2}(x_1 - c_1)(x_2 - c_2) + \cdots \]

Taylor expansion at \(x_1 = c_1\) and \(x_2 = c_2\)

- One-dimensional Taylor Series Expansion

\[ y(x_1, c_2) = y(c_1, c_2) + \frac{\partial y(c_1, c_2)}{\partial x_1}(x_1 - c_1) + \frac{\partial^2 y(c_1, c_2)}{\partial x_1^2}(x_1 - c_1)^2 + \cdots \]

Taylor expansion at \(x_1 = c_1\)

\[ y(c_1, x_2) = g(c_1, c_2) + \frac{\partial y(c_1, c_2)}{\partial x_2}(x_2 - c_2) + \frac{\partial^2 y(c_1, c_2)}{\partial x_2^2}(x_2 - c_2)^2 + \cdots \]

Taylor expansion at \(x_2 = c_2\)

(Li et al., 2001)
**Convergence Issue**

- **Two-dimensional Taylor Series Expansion**

\[ y(x_1, x_2) = y(c_1, c_2) + \frac{\partial y(c_1, c_2)}{\partial x_1}(x_1 - c_1) + \frac{\partial y(c_1, c_2)}{\partial x_2}(x_2 - c_2) + \frac{\partial^2 y(c_1, c_2)}{\partial x_1^2}(x_1 - c_1)^2 \]

\[ + \frac{\partial^2 y(c_1, c_2)}{\partial x_2^2}(x_2 - c_2)^2 + \frac{\partial^2 y(c_1, c_2)}{\partial x_1 \partial x_2}(x_1 - c_1)(x_2 - c_2) + \cdots \]

2D cooperative effect

- **Sum of Two One-dimensional Taylor Series**

\[ y(x_1, c_2) + y(c_1, x_2) - y(c_1, c_2) = y(c_1, c_2) + \frac{\partial y(c_1, c_2)}{\partial x_1}(x_1 - c_1) + \frac{\partial y(c_1, c_2)}{\partial x_2}(x_2 - c_2) \]

\[ + \frac{\partial^2 y(c_1, c_2)}{\partial x_1^2}(x_1 - c_1)^2 + \frac{\partial^2 y(c_1, c_2)}{\partial x_2^2}(x_2 - c_2)^2 + \cdots \]

(Li et al., 2001)
Errors in HDMR Approximation

- Residual Error

\[ y(x) - \hat{y}(x) = \sum_{j_2}^{\infty} \sum_{j_1}^{\infty} \frac{1}{j_1! j_2!} \sum_{i_1 < i_2} \frac{\partial^{j_1+j_2}}{\partial x_{i_1}^{j_1} \partial x_{i_2}^{j_2}} y(c) (x_{i_1} - c_{i_1})^{j_1} (x_{i_2} - c_{i_2})^{j_2} \]

\[ y(x) = \hat{y}(x) \]

- \( \hat{y}(x) \) represents reduced dimensional approximation, because only \( N \) number of 1-dimensional model approximation are required, as opposed to one \( N \)-dimensional approximation in \( y(x) \).

- If higher partial derivatives are negligibly small, \( \hat{y}(x) \) provides a convenient approximation of \( y(x) \)

- First-order HDMR expansion is the sum of all Taylor series terms, which contains only variable \( x_i \). Similarly, second-order HDMR expansion is the sum of all Taylor series terms, which contains only variable \( x_i \) and \( x_j \). Therefore any truncated HDMR expansion provides better approximation of \( y(x) \) than any truncated Taylor series (e.g., FORM/SORM).

(Li et al., 2001)
HDMR (Continued)

First-order Approximation

\[ \hat{y}_1(x) \approx \sum_{i=1}^{N} \sum_{j=1}^{n} \phi_j(x_i) y(c_1, \cdots, c_{i-1}, x^{(j)}_i, c_{i+1}, \cdots, c_N) - (N - 1)y(c) \]

Reference point

Interpolation function

One Variable

Two Variable
HDMR (Continued)

Second-order Approximation

\[ \hat{y}_2(x) \approx \sum_{i_1, i_2 = 1}^{N} \sum_{j_1 = 1}^{n} \sum_{j_2 = 1}^{n} \phi_{j_1 j_2}(x_{i_1}, x_{i_2}) y(c_1, \cdots, c_{i_1-1}, x_{i_1}^{(j_1)}, c_{i_1+1}, \cdots, c_{i_2-1}, x_{i_2}^{(j_2)}, c_{i_2+1}, \cdots, c_N) \]

- \((N-2)\sum_{i=1}^{N} \sum_{j=1}^{n} \phi_j(x_i) y(c_1, \cdots, c_{i-1}, x_i^{(j)}, c_{i+1}, \cdots, c_N)\) + \frac{(N-1)(N-2)}{2} y(c)

Two Variables

x_1

x_2
HDMR (Continued)

- **Computational Effort (Calculating Coefficients)**

  No. of FEA for a linear/nonlinear problem,

\[
y(c) \rightarrow 1 \text{ FEA}
\]

\[
y(c_1, \cdots, c_{i-1}, x_i^{(j)}, c_{i+1}, \cdots, c_N) \\
(i = 1, \cdots, N; j = 1, \cdots, n)
\]

\[
y(c_1, \cdots, c_{i-1}, x_i^{(j_1)}, c_{i+1}, \cdots, c_{i_2-1}, x_{i_2}^{(j_2)}, c_{i_2+1}, \cdots, c_N) \\
(i_1, i_2 = 1, \cdots, N; j_1, j_2 = 1, \cdots, n)
\]

\[
1 \rightarrow nN \text{ FEA}
\]

\[
n(N-1)n^2/2 \rightarrow N(N-1)(n-1)/2 + (n-1)N + 1 \text{ (quadratic)}
\]

First-order: \((n-1)N + 1\) (linear)

Second-order: \(N(N-1)(n-1)^2/2 + (n-1)N + 1\) (quadratic)

(Chowdhury & Rao, 2009)
Example 1: 2-DOF system

\[ m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \]

\[ k(x) = \begin{bmatrix} k_1(x) + k_3 & -k_3 \\ -k_3 & k_2(x) + k_3 \end{bmatrix} \]

\[
\begin{align*}
m_1 &= 1.0 \text{ kg} \\
m_2 &= 1.5 \text{ kg} \\
k_1(x) &= 1000(1 + 0.25x_1) \text{ N/m} \\
k_2(x) &= 1100(1 + 0.25x_2) \text{ N/m} \\
k_1(x) &= 100 \text{ N/m} \\
x &= \{x_1, x_2\}^T \\
\mu_x &= 0, \Sigma_x = I
\end{align*}
\]
Example 1: 2-DOF system

Exact eigenvalues for the 2-DOF system
Example 1: 2-DOF system

Probability densities for the 2-DOF system
Example 1: 2-DOF system

Scattered plot of $\lambda_1$ & $\lambda_2$

$\rho_{12} = 0.312$

$\rho_{12} = 0.293$

$\rho_{12} = 0.295$
Example 2: 3-DOF system
Example 2: 3-DOF system

\[ M(x) = \begin{bmatrix} m_1(x) & 0 & 0 \\ 0 & m_2(x) & 0 \\ 0 & 0 & m_3(x) \end{bmatrix} \]

\[ k(x) = \begin{bmatrix} k_1(x) + k_4(x) + k_6(x) & -k_4(x) & -k_6(x) \\ -k_4(x) & k_4(x) + k_5(x) + k_2(x) & -k_5(x) \\ -k_6(x) & -k_5(x) & k_5(x) + k_3(x) + k_6(x) \end{bmatrix} \]

\[ m_i(x) = \mu_i x_i; \quad i = 1, 2, 3 \quad \text{with} \quad \mu_i = 1.0 \text{kg}; \quad i = 1, 2, 3 \]

\[ k_i(x) = \mu_{i+3} x_{i+3}; \quad i = 1, \ldots, 6 \quad \text{with} \]

\[ \mu_{i+3} = 1.0 \text{N/m}; i = 1, \ldots, 5 \]

\[ \mu_9 = 3.0 \text{N/m (Case 1 & Case 3)}; \mu_9 = 1.275 \text{N/m (Case 2)} \]

\[ x = \{x_1, \ldots, x_9\}^T; \quad \boldsymbol{\mu} = 0, \sum_x = \nu^2 I \]

\[ \nu = 0.15 \text{(Case 1 & Case 2)}; \]

\[ \nu = 0.30 \text{(Case 3)}; \]
Case 1: Well separated eigenvalues

Probability densities for the 3-DOF system
Case 1: Well separated eigenvalues

Scattered plot of $\lambda_1$ & $\lambda_2$
Case 1: Well separated eigenvalues

Scattered plot of $\lambda_1$ & $\lambda_3$
Case 1: Well separated eigenvalues

Scattered plot of $\lambda_2$ & $\lambda_3$

$\rho_{23} = 0.234$

$\rho_{23} = 0.221$

$\rho_{23} = 0.232$
Case 2: Closely spaced eigenvalues

Probability densities for the 3-DOF system
Case 3: Large statistical variation of input

Probability densities for the 3-DOF system
Conclusions

- The statistics of the eigenvalues of linear stochastic dynamic systems has been considered.
- HDMR approximation method has been developed for efficient scheme for random eigenvalue problems.
- PDF of the eigenvalues are obtained using the maximum entropy method.
- Yields accurate and convergent solutions.
- Future works will look into joint moments and PDF of the eigenvalues and eigenvectors.
References


Thank you