Reliability approximations via asymptotic distribution

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Outline of the presentation

- Introduction to structural reliability analysis
- Limitation of FORM/SORM in high dimensions
- Asymptotic distribution of quadratic forms
- Strict asymptotic formulation
- Weak asymptotic formulation
- Numerical results
- Conclusions & discussions
Structural reliability analysis

Probability of failure

\[ P_f = (2\pi)^{-n/2} \int_{g(x) \leq 0} e^{-\frac{x^T x}{2}} dx \]

**x ∈ ℝ^n**: Gaussian parameter vector

**g(x)**: failure surface

Maximum contribution comes from the neighborhood where \( x^T x / 2 \) is minimum subject to \( g(x) \leq 0 \). The design point \( x^* \):

\[ x^* : \min\{(x^T x)/2\} \quad \text{subject to} \quad g(x) = 0. \]
Graphical explanation

Failure domain
\[ g(x) \leq 0 \]

Actual failure surface
\[ g(x) = 0 \]

SORM approximation
\[ y_n = \beta + y^T Ay \]

FORM approximation
\[ y_n = \beta \]

\[ \frac{x^*}{\beta} = -\frac{\nabla g}{|\nabla g|} = \alpha^* \]
FORM/SORM approximations

\[ P_f \approx \text{Prob} \left[ y_n \geq \beta + y^T A y \right] = \text{Prob} \left[ y_n \geq \beta + U \right] \quad (1) \]

where

\[ U : \mathbb{R}^{n-1} \mapsto \mathbb{R} = y^T A y, \]

is a quadratic form in Gaussian random variable. The eigenvalues of \( A \), say \( a_j \), can be related to the principal curvatures of the surface \( \kappa_j \) as \( a_j = \kappa_j / 2 \).

Considering \( A = O \) in Eq. (1), we have the FORM:

\[ P_f \approx \Phi(-\beta) \]
SORM approximations

Breitung’s asymptotic formula (1984):

\[ P_f \to \Phi(-\beta) \left\| \mathbf{I}_{n-1} + 2\beta \mathbf{A} \right\|^{-1/2} \quad \text{when} \quad \beta \to \infty \]

Hohenbichler and Rackwitz’s improved formula (1988):

\[ P_f \approx \Phi(-\beta) \left\| \mathbf{I}_{n-1} + 2 \frac{\varphi(\beta)}{\Phi(-\beta)} \mathbf{A} \right\|^{-1/2} \]
Numerical example

Consider a problem for which the failure surface is exactly parabolic: \[ g = -y_n + \beta + y^T A y \]

- We choose \( n \) and the value of \( \text{Trace} (A) \).
- When \( \text{Trace} (A) = 0 \) the failure surface is effectively linear. Therefore, the more the value of \( \text{Trace} (A) \), the more non-linear the failure surface becomes.
- It is assumed that the eigenvalues of \( A \) are uniform random numbers.
$P_f$ for small $n$

Failure probability for $n - 1 = 3$, $\text{Trace}(A) = 1$
$P_f$ for large $n$

Failure probability for $n - 1 = 100$, Trace ($A$) = 1

Asymptotic: $\beta \rightarrow \infty$ (Breitung, 84)
Hohenbichler & Rackwitz, 88
Exact (MCS)
The problem with high-dimension

- If \( n \), i.e. the dimension is large, the computation time to obtain \( P_f \) using any tools will be high.
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- **Question 1**: Suppose we have followed the ‘normal route’ and obtained $x^*$, $\beta$ and $A$. Why the results from classical FORM/SORM is not satisfactory in a high dimensional problem?

- **Question 2**: What is a ‘high dimension’?
The problem with high-dimension

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- **Question 1:** Suppose we have followed the ‘normal route’ and obtained \( x^*, \beta \) and \( A \). Why the results from classical FORM/SORM is not satisfactory in a high dimensional problem?

- **Question 2:** What is a ‘high dimension’?

- Only simulation methods (Au & Beck, 2003; Koutsourelakis et al., 2004) are available at present for problems with high dimension.
Asymptotic distribution of quadratic forms

Moment generating function:

\[
M_U(s) = \|\mathbf{I}_{n-1} - 2s\mathbf{A}\|^{-1/2} = \prod_{k=1}^{n-1} (1 - 2sa_k)^{-1/2}
\]

Now construct a sequence of new random variables \( q = \frac{U}{\sqrt{n}} \). The moment generating function of \( q \):

\[
M_q(s) = M_U\left(\frac{s}{\sqrt{n}}\right) = \prod_{k=1}^{n-1} \left(1 - 2sa_k/\sqrt{n}\right)^{-1/2}
\]
Asymptotic distribution

Truncating the Taylor series expansion:

\[ \ln(M_q(s)) \approx \text{Trace}(A) \frac{s}{\sqrt{n}} + \left(2 \text{Trace}(A^2)\right) \frac{s^2}{2n} \]

We assume \( n \) is large such that the following conditions hold

\[ \frac{2}{n} \text{Trace}(A^2) < \infty \]

and

\[ \frac{2^r}{n^{r/2}} \text{Trace}(A^r) \to 0, \forall r \geq 3 \]
Therefore, the moment generating function of 
$U = q\sqrt{n}$ can be approximated by:

$$M_U(s) \approx e^{\text{Trace}(A)s + \left(2\text{Trace}(A^2)\right)s^2/2}$$

From the uniqueness of the Laplace Transform pair it follows that $U$ asymptotically approaches a Gaussian random variable with mean $\text{Trace}(A)$ and variance $2\text{Trace}(A^2)$, that is

$$U \sim N_1\left(\text{Trace}(A), 2\text{Trace}(A^2)\right) \quad \text{when} \quad n \to \infty$$
Minimum number of random variables

The error in neglecting higher order terms:

\[
\frac{1}{r} \left( \frac{2s}{\sqrt{n}} \right)^r \text{Trace} \left( A^r \right), \quad \text{for } r \geq 3.
\]

Using \( s = \beta \) and assuming there exist a small real number \( \epsilon \) (the error) we have

\[
\frac{1}{r} \frac{(2\beta)^r}{n^{r/2}} \text{Trace} \left( A^r \right) < \epsilon \quad \text{or} \quad n > \frac{4\beta^2}{\sqrt{r^2 \epsilon^2}} \left( \sqrt{\text{Trace} \left( A^r \right)} \right)^2
\]
We rewrite (1):

\[ P_f \approx \text{Prob} \left[ y_n \geq \beta + U \right] = \text{Prob} \left[ y_n - U \geq \beta \right] \]

Since \( U \) is asymptotically Gaussian, the variable \( z = y_n - U \) is also Gaussian with mean \((-\text{Trace}(A))\) and variance \((1 + 2 \text{Trace}(A^2))\). Thus,

\[ P_{f_{\text{Strict}}} \rightarrow \Phi(-\beta_1), \quad \beta_1 = \frac{\beta + \text{Trace}(A)}{\sqrt{1 + 2 \text{Trace}(A^2)}}, \quad n \rightarrow \infty \]
**Graphical explanation**

\[ m = \text{Trace}(A), \quad \sigma^2 = 2\text{Trace}(A^2) \]

Failure surface: \( y_n - U \geq \beta \). Using the standardizing transformation \( Y = (U - m)/\sigma \), modified failure surface

\[
\frac{y_n}{\beta + m} + \frac{Y}{\beta + m} \geq 1.
\]

From \( \triangle AOB \),

\[
\sin \theta = \frac{\tan \theta}{\sqrt{1 + \tan^2 \theta}} = \frac{\sigma}{\sqrt{1 + \sigma^2}}.
\]

Therefore, from \( \triangle OBY^* \):

\[
\beta_1 = \frac{\beta + m}{\sigma} \sin \theta = \frac{\beta + m}{\sqrt{1 + \sigma^2}} = \frac{\beta + \text{Trace}(A)}{\sqrt{1 + 2\text{Trace}(A^2)}}.
\]

If \( n \) is small, \( m, \sigma \) will be small. When \( m, \sigma \rightarrow 0 \), \( AB \) rotates clockwise and eventually becomes parallel to the Y-axis with a shift of +\( \beta \). In this situation \( y^* \rightarrow x^* \) in the \( y_n \)-axis and \( \beta_1 \rightarrow \beta \) as expected. This explains why classical F/SORM approximations based on the original design point \( x^* \) do not work well when a large number of random variables are considered.
Weak asymptotic formulation

\[ P_f \approx \text{Prob} \left[ y_n \geq \beta + U \right] \]

\[ = \int_{\mathbb{R}} \left\{ \int_{\beta + u}^{\infty} \varphi(y_n) dy_n \right\} p_U(u) du = E \left[ \Phi(-\beta - U) \right] \]

Noticing that \( u \in \mathbb{R}^+ \) as \( A \) is positive definite we rewrite

\[ P_f \approx \int_{\mathbb{R}^+} e^{\ln[\Phi(-\beta-u)] + \ln[p_U(u)]} du \]
Weak asymptotic formulation

For the maxima of the integrand (say at point $u^*$)

$$\frac{\partial}{\partial u} \{ \ln [\Phi(-\beta - u)] + \ln [p_U(u)] \} = 0$$

Recalling that

$$p_U(u) = (2\pi)^{-1/2} \sigma^{-1} e^{-(u-m)^2/(2\sigma^2)}$$

we have

$$\frac{\varphi(\beta + u)}{\Phi(-(\beta + u))} = \frac{m - u}{\sigma^2}$$
Weak asymptotic formulation

After some simplifications, the failure probability using weak asymptotic formulation:

\[
P_{f_{\text{Weak}}} \rightarrow \Phi (-\beta_2) e^{-\left(2\beta_2^2 \text{Trace}(A^2) - \beta_2 \text{Trace}(A)\right)} \frac{\beta + \text{Trace}(A)}{1 + 2 \text{Trace}(A^2)},
\]

where \( \beta_2 = \frac{\beta + \text{Trace}(A)}{1 + 2 \text{Trace}(A^2)} \) when \( n \rightarrow \infty \)

For the small \( n \) case, neglecting the ‘trace effect’ it can be seen that \( P_{f_{\text{Weak}}} \) approaches to Breitung’s formula.
Comparison of $P_f$

Asymptotic: $\beta \to \infty$ (Breitung, 84)

Hohenbichler & Rackwitz, 88

Strict asymptotic, $n \to \infty$

Weak asymptotic, $n \to \infty$

Exact (MCS)

Failure probability for $\text{Trace}(A) = 1$, $\beta = 3$
Comparison of $P_f$

Failure probability for $\text{Trace} (A) = 1, \beta = 4$
Comparison of $P_f$

Asymptotic: $\beta \rightarrow \infty$ (Breitung, 84)

Hohenbichler & Rackwitz, 88

Strict asymptotic, $n \rightarrow \infty$

Weak asymptotic, $n \rightarrow \infty$

Exact (MCS)

Failure probability for $\text{Trace } (A) = 1$, $\beta = 5$
Comparison of $P_f$

Failure probability for $\text{Trace}(A) = 1, \beta = 6$
Summary & conclusions

- Geometric analysis shows that the classical design point should be modified in high dimension. This also explains why classical FORM/SORM work poorly in high dimension.
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- Geometric analysis shows that the classical design point should be modified in high dimension. This also explains why classical FORM/SORM work poorly in high dimension.

- In the context of classical FORM/SORM, the number of random variables $n$ can be considered as large if

$$n > \frac{4\beta^2}{\sqrt[3]{9\epsilon^2}} \left( \sqrt[3]{\text{Trace} \left( A^3 \right)} \right)^2$$
$P_{f_{\text{strict}}} \rightarrow \Phi (-\beta_1), \quad \beta_1 = \frac{\beta + \text{Trace}(\mathbf{A})}{\sqrt{1 + 2 \text{Trace}(\mathbf{A}^2)}}, \quad n \rightarrow \infty$

The strict asymptotic formula can be viewed as the ‘correction’ needed to the existing FORM formula in high dimension.
Summary & conclusions

\[
P_{f_{\text{Weak}}} \rightarrow \Phi(-\beta_2) e^{-\left(2\beta_2^2 \text{Trace}(A^2) - \beta_2 \text{Trace}(A)\right)} \frac{\sqrt{\left\|\mathbf{I}_{n-1} + 2\beta_2 \mathbf{A}\right\|}}{\sqrt{\left\|\mathbf{I}_{n-1} + 2\beta_2 \mathbf{A}\right\|}},
\]

where \( \beta_2 = \frac{\beta + \text{Trace}(\mathbf{A})}{1 + 2 \text{Trace}(\mathbf{A}^2)} \) when \( n \to \infty \)

The weak asymptotic formula can viewed as the correction needed to the existing SORM formula in high dimension.
References
