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Fourier representation of random media fields in stochastic finite element modelling

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Abstract

Purpose – To provide an explicit representation for wide-sense stationary stochastic fields which can be used in stochastic finite element modelling to describe random material properties.

Design/methodology/approach – This method represents wide-sense stationary stochastic fields in terms of multiple Fourier series and a vector of mutually uncorrelated random variables, which are obtained by minimizing the mean-squared error of a characteristic equation and solving a standard algebraic eigenvalue problem. The result can be treated as a semi-analytic solution of the Karhunen-Loëve expansion.

Findings – According to the Karhunen-Loëve theorem, a second-order stochastic field can be decomposed into a random part and a deterministic part. Owing to the harmonic essence of wide-sense stationary stochastic fields, the decomposition can be effectively obtained with the assistance of multiple Fourier series.

Practical implications – The proposed explicit representation of wide-sense stationary stochastic fields is accurate, efficient and independent of the real shape of the random structure in consideration. Therefore, it can be readily applied in a variety of stochastic finite element formulations to describe random material properties.

Originality/value – This paper discloses the connection between the spectral representation theory of wide-sense stationary stochastic fields and the Karhunen-Loëve theorem of general second-order stochastic fields, and obtains a Fourier-Karhunen-Loëve representation for the former stochastic fields.

Keywords Finite element analysis, Modelling

Paper type Research paper



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1. Introduction

1.1 Background

The finite element method, which has become an essential numerical tool in numerous engineering and scientific areas, is normally based on a deterministic model. In other words, all the system parameters, such as loads, geometric configurations and material properties, have to be known in order to perform a successful analysis. In reality, however, the treatment differs due to the existence of various unpredictable and immeasurable factors. In a considerable number of engineering and scientific problems in which the scale of random fluctuation is relatively small, the deterministic model

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provides a good approximation. However, it is often extremely difficult, if not impossible, to obtain an exhaustive deterministic description of a system. Consider such cases as soils, rocks, ground water flows, structures with significant uncertain parameters, and micro-scale materials; in the event of these cases arising, a deterministic model may cease to be useful. Hence, stochastic finite element methods (SFEM), which take into consideration various uncertainties in the finite element formulation, are required for modelling random phenomena. As shown in Figure 1, the research of SFEMs has become of increasing interest in recent years (Shinozuka, 1972; Yamazaki *et al.*, 1988; Papadrakakis and Kotsopoulos, 1999; Charmpis and Papadrakakis, 2005; Pradlwarter *et al.*, 2005; Liu *et al.*, 1986a, b; Kamiński and Kleiber, 1996; Hien and Kleiber, 1997; Ghanem and Spanos, 1991; Babuška and Chatzipantelidis, 2002; Frauenfelder *et al.*, 2005; Vanmarcke and Grigoriu, 1983; Li and Derkiureghian, 1993; Harada and Shinozuka, 1986; Li *et al.*, n.d.).

While there are various approaches in formulating a SFEM, the three most prominent methods are:

- (1) The Monte Carlo method (Shinozuka, 1972; Yamazaki *et al.*, 1988; Papadrakakis and Kotsopoulos, 1999; Charmpis and Papadrakakis, 2005; Pradlwarter *et al.*, 2005). This method first generates a set of sample paths for the given random medium, then, for each sample path, a standard deterministic simulation is performed to obtain the corresponding solution sample. From the solution samples the empirical probability distribution and empirical statistical quantities of the random solution can be calculated. Until recently, it appears that only the Monte Carlo method, inclusive of improvements, has been widely accepted as a versatile and practical approach in many engineering situations, such as dynamic reliability assessment of general systems (Pradlwarter *et al.*, 2005). However, the Monte Carlo method is generally attributed as being computationally expensive.

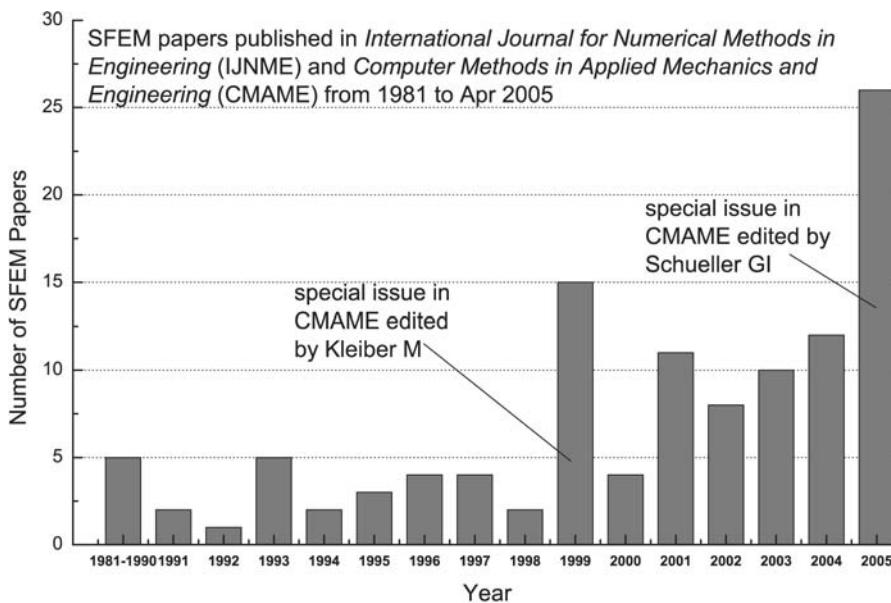


Figure 1.
SFEM papers published in *International Journal for Numerical Methods in Engineering* and *Computer Methods in Applied Mechanics and Engineering* from 1981 to 2005

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- (2) The perturbation method (Liu *et al.*, 1986a, b; Kami ski and Kleiber, 1996; Hien and Kleiber, 1997). The method commences with an expansion of a Taylor's series. The expansion is performed with respect to the random variables that represent the given stochastic field of random media. The unknown coefficients in the Taylor expansion are then obtained from grouping like polynomials, where upon the sum of these like polynomials is set to zero. The perturbation method is computationally more efficient than the Monte Carlo method. The failing of this method is a dependence on the random fluctuations being small. Another disadvantage of the perturbation method is that it mainly focuses on the second-order estimate of the response (e.g. variance and covariance) and does not permit higher-order statistical estimates.
- (3) The polynomial chaos expansion method (Ghanem and Spanos, 1991; Babuška and Chatzipantelidis, 2002; Frauenfelder *et al.*, 2005). This method expands the unknown stochastic field with multiple Hermite polynomials of random variables, i.e. polynomial chaos, and solves the associated unknown coefficients through a Galerkin approach. There is no limit to the scale of random fluctuations and the polynomial chaos expansion can, at least in principle, approximate a functional of Gaussian random variables to any accuracy. However, the polynomial chaos expansion method can only be rigorously applied to problems that merely consist of Gaussian random variables. In addition, it does become increasingly difficult to derive and code multiple Hermite polynomials (the basic building blocks of this method) as the number of random variables increases, and the associated computational cost also increases significantly. In fact, the polynomial chaos expansion has hardly been applied to cases with more than ten random variables.

Hence, compared with the well developed finite element method, the SFEM is still in its infancy, and further development is needed to advance the SFEM to a level where both reasonable mathematical rigour and sufficient computational efficiency are achieved so that the SFEM can be applied to solve large-scale practical problems.

A key issue shared by all the SFEMs mentioned above is how to represent the random variation of material properties through the medium, for example, random Young's modulus, random Poisson's ratio and dispersive random defects, etc. The governing equations of physical systems require an explicit definition of the stochastic fields of random material properties in terms of a series of random variables. This is, however, only possible for very simple cases or when significant simplification can be made (Papadrakakis and Kotsopoulos, 1999; Charmpis and Papadrakakis, 2005). Let $X(\tau)$ denote the stochastic field of a certain random material property, in a practical problem $X(\tau)$ is often defined implicitly by its first- and second-order statistical moments, i.e. the expectation function $m(\tau)$ and the covariance function $\text{Cov}_X(\tau_1, \tau_2)$. Consequently, the aim of random-material-property representation is to obtain an explicit approximation of $X(\tau)$, $Y(\tau)$:

$$Y(\tau) = \sum_{i=1}^M y_i p_i(\tau) \quad (1)$$

where $\{y_1, y_2, \dots, y_M\}$ denotes a finite sequence of random variables corresponding to a family of deterministic functions $\{p_1(\tau), p_2(\tau), \dots, p_M(\tau)\}$. Naturally, it is required that $Y(\tau)$ satisfies, in a certain sense:

$$E(Y(\tau)) \approx m(\tau) \quad (2)$$

and:

$$\text{Cov}(Y(\tau_1), Y(\tau_2)) \approx \text{Cov}_X(\tau_1, \tau_2) \quad (3)$$

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fields

where $E(\bullet)$ denotes the expectation of a random variable and $\text{Cov}(\bullet, \bullet)$ the covariance between two random variables. It should be noted that only Gaussian stochastic fields are completely described by the expectation and covariance functions, and higher-order statistical moments are required to fully determine a non-Gaussian stochastic field. Nevertheless, the expectation and the covariance functions provide not only the most basic information of stochastic fields, but also the only reliable input information available in many practical situations where sample data are limited.

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1.2 Overview on stochastic field representation

The solution to the stochastic-field-representation problem (1) involves answering two questions:

- (1) how to choose the deterministic function family $p_i(\tau) i = 1, 2, \dots, M$; and
- (2) by which criterion to approximate the stochastic field $X(\tau)$ such that equations (2) and (3) are justified in a certain measure.

In regard to the first question, piecewise polynomials (associated with finite element meshes) have been extensively used in the literature (Shinozuka, 1972; Yamazaki *et al.*, 1988; Papadrakakis and Kotsopoulos, 1999; Charmpis and Papadrakakis, 2005; Pradlwarter *et al.*, 2005; Liu *et al.*, 1986a, b; Kamiński and Kleiber, 1996; Hien and Kleiber, 1997; Ghanem and Spanos, 1991; Babuška and Chatzipantelidis, 2002; Frauenfelder *et al.*, 2005; Vanmarcke and Grigoriu, 1983; Li and Derkiureghian, 1993; Harada and Shinozuka, 1986). The advantages of taking piecewise polynomials as the deterministic function basis include the flexibility for different domains and compatibility with the standard finite element method. The disadvantages of an FE-mesh basis are its poor adaptability to small correlation distances, i.e. a fine mesh is required when the correlation distance is small, and high computational costs especially for 3D cases.

For the second question, various criteria have been employed in different applications, mainly including:

- The middle point scheme (Shinozuka, 1972; Yamazaki *et al.*, 1988) in which the stochastic field within an element is represented by a random variable located at the centre of the corresponding element, and the samples of these central random variables are generated with the assistance of the Cholesky decomposition of their covariance matrix.
- The local average scheme (Vanmarcke and Grigoriu, 1983) that represents the one-dimensional stochastic field within a beam element by the spatial average of the corresponding element.
- The shape function scheme (Liu *et al.*, 1986a, b; Kamiński and Kleiber, 1996; Hien and Kleiber, 1997) that interpolates the stochastic field within an element by nodal random variables and the associated shape functions, and enforces the covariance between each pair of nodes to take the exact value from the given covariance function. However, in each element, there is no link between the interpolated stochastic field and the real one.

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- The K-L expansion scheme (Ghanem and Spanos, 1991; Babuška and Chatzipantelidis, 2002; Frauenfelder *et al.*, 2005) that approximates the stochastic field according to the K-L theorem (see Section 2 for details).
 - The direct least-squares approximation scheme (Li and Derkiureghian, 1993) in which the approximation accuracy within an element is measured by the variance of the error between the real stochastic field and the approximate one.

The first three criteria give only very approximate estimations of the stochastic field under consideration. The K-L expansion (Loève, 1977) was first introduced into the SFEM community by Ghanem and Spanos (1991) to approximate Gaussian stochastic fields, and is viewed as one of the most significant developments towards the improvement of mathematical rigorousness of the SFEM. Although this form of expansion has received considerable attention, further applications are limited due to the lack of an efficient and accurate method with which to solve the inherent characteristic equation. The difficulties are evident when considering a large 3D domain or small correlation distances. Both theoretical analysis and the finite element method have been employed to solve the resulting characteristic equation; however, effectiveness, efficiency and accuracy are not always satisfied (Li and Derkiureghian, 1993). For example, (Frauenfelder *et al.*, 2005), it is difficult, especially in 3D cases, to reduce the approximation errors of the FE-mesh-based K-L expansion within reasonable computational costs. Consequently, a least-squares solution based on a finite element mesh is presented in Li and Derkiureghian (1993) to approximate stochastic fields. This direct least-squares approximation scheme is more accurate and computationally efficient than the FE-mesh-based K-L expansion scheme, but the unique properties of the K-L expansion are not retained.

The main contribution of the current work is to propose a more accurate and computationally more efficient method for the explicit representation of wide-sense stationary stochastic fields (Loève, 1977; Yaglom, 2004) which are extensively used in SFEM formulations (Shinozuka, 1972; Yamazaki *et al.*, 1988; Papadrakakis and Kotsopoulos, 1999; Charmpis and Papadrakakis, 2005; Pradlwarter *et al.*, 2005; Liu *et al.*, 1986a, b; Kamiński and Kleiber, 1996; Hien and Kleiber, 1997; Ghanem and Spanos, 1991; Babuška and Chatzipantelidis, 2002; Frauenfelder *et al.*, 2005; Vanmarcke and Grigoriu, 1983; Li and Derkiureghian, 1993; Harada and Shinozuka, 1986; Li *et al.*, n.d.) to describe random material properties. More specifically, multiple Fourier series instead of piecewise polynomials are chosen as the deterministic function basis and the optimal approximation in a mean-square sense is obtained by solving a standard algebraic eigenvalue problem which is accurately constructed through an explicit procedure. This new method can be viewed as a semi-analytic solution of the K-L expansion whose desirable properties are all retained, and is also sufficiently versatile so that the performance is not sensitive to the correlation distance or the dimensions of the given stochastic field.

The remainder of this paper is organized as follows. Preliminary knowledge for the proposed method is outlined in the first two sections: Section 2 explains the formulation of the K-L expansion along with its underlying criterion by which the stochastic field is approximated; Section 3 addresses the format and several useful properties of multiple Fourier series. In Section 4, an explicit procedure to obtain the optimal approximation of a stochastic field is presented in detail. Then, three numerical

examples are examined in Section 5 to verify the efficiency and accuracy of this new stochastic-field-representation solution. The paper concludes in Section 6 with a summary of the main features of the proposed method and recommendations for its use in practical problems.

Symbols \mathbb{R} , \mathbb{Z} and \mathbb{N} in this paper are exclusively used with their standard meanings, i.e. \mathbb{R} denotes the real number set, $\mathbb{Z} \triangleq \{0, \pm 1, \pm 2, \dots\}$ the integer set and $\mathbb{N} \triangleq \{1, 2, 3, \dots\}$ the natural number set. A vector always indicates a column vector unless specifically declared otherwise.

2. The Karhunen-Loëve expansion

2.1 Mercer's theorem

Definition 1. A continuous real function $R(s, t)$ $s, t \in [a, b]$ is continuous symmetric non-negative definite if and only if for $\forall (s, t) \in [a, b] \times [a, b]$ there is $R(s, t) = R(t, s)$, and $\sum_{i=1}^m \sum_{j=1}^m R(t_i, t_j) c_i c_j \geq 0$ for all finite sequences of points t_1, \dots, t_m of $[a, b]$ and all choices of non-zero real vector $c = (c_1, \dots, c_m)$.

Theorem 1. Suppose that $R(s, t)$ is continuous symmetric non-negative definite in the interval $[a, b]$. Then there exists a set of orthonormal functions $\{e_i(t)\}_{i \in \mathbb{N}}$ in $L^2([a, b])$ satisfying the following characteristic equation:

$$\int_a^b R(s, t) e(s) ds = \lambda e(t) \quad (4)$$

where the eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$ are non-negative and the eigen-functions $\{e_i(t)\}_{i \in \mathbb{N}}$ corresponding to non-zero eigenvalues are continuous in $[a, b]$. Furthermore, the integral kernel $R(s, t)$ has the following representation:

$$R(s, t) = \sum_{i=1}^{\infty} \lambda_i e_i(s) e_i(t) \quad (5)$$

whose convergence is absolute and uniform. In particular:

$$\text{Tr}(R(s, t)) \triangleq \int_a^b R(t, t) dt = \sum_{i=1}^{\infty} \lambda_i. \quad (6)$$

Theorem 1 is known as Mercer's theorem, one of the most important theoretical tools in the theory of integral equations, and equality (6) is termed the trace relation (Zaanen, 1964; Courant and Hilbert, 1953; Simon, 1979).

2.2 The K-L expansion

In the Hilbert space theory of stochastic processes, the K-L expansion is a technique that can be used to simplify a stochastic process; more formally it is a transformation that chooses a new set of basis functions for the stochastic process such that the greatest variance by any projection of the stochastic process lies on the first basis-function axis (called the first principal component); the second greatest variance on the second axis, and so on. Assuming a zero mean, the first principal component $e_1(t)$ of a stochastic process $X(t)$ $t \in [a, b]$ can be defined as:

$$e_1(t) = \arg \max_{\int_a^b e^2(t) dt = 1} E \left\{ \left[\int_a^b X(t) e(t) dt \right]^2 \right\} \quad (7)$$

where $\arg \max$ stands for the argument of the maximum. When the first $k - 1$ components are defined, the k th component can be found by subtracting the first $k - 1$ principal components from $X(t)$:

$$\hat{X}_{k-1}(t) = X(t) - \sum_{i=1}^{k-1} e_i(t) \int_a^b X(t) e_i(t) dt \quad (8)$$

and then by treating this as the new stochastic process to find its principal component:

$$e_k(t) = \arg \max_{\int_a^b e^2(t) dt = 1} E \left\{ \left[\int_a^b \hat{X}_{k-1}(t) e(t) dt \right]^2 \right\} \quad (9)$$

The stochastic process $X(t)$ then has the following representation:

$$X(t) = \sum_{i=1}^{\infty} z_i e_i(t) \quad (10)$$

where z_i are random variables satisfying:

$$z_i = \int_a^b X(t) e_i(t) dt \quad (11)$$

For a centred continuous real-valued stochastic process $X(t)$ $t \in [a, b]$ (where centred means that the expectation function $E(X(t))$ is equal to zero for all t), consider its covariance function:

$$\begin{aligned} \text{Cov}_X(t_1, t_2) &= \text{Cov}(X(t_1), X(t_2)) \\ &= E\{[X(t_1) - E(X(t_1))][X(t_2) - E(X(t_2))]\} \quad t_1, t_2 \in [a, b] \\ &= E(X(t_1)X(t_2)) \end{aligned} \quad (12)$$

It is obvious that the above covariance function is continuous and symmetric. Let t_1, \dots, t_m denote m points in $[a, b]$ and $c = (c_1, \dots, c_m)$ a non-zero real vector, the non-negative definite property follows immediately from:

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^m \text{Cov}_X(t_i, t_j) c_i c_j &= \sum_{i=1}^m \sum_{j=1}^m E(X(t_i)X(t_j)) c_i c_j \\ &= \sum_{i=1}^m \sum_{j=1}^m E(c_i X(t_i) c_j X(t_j)) \\ &= E \left(\sum_{i=1}^m \sum_{j=1}^m c_i X(t_i) c_j X(t_j) \right) \\ &= E \left(\left(\sum_{i=1}^m c_i X(t_i) \right)^2 \right) \geq 0 \end{aligned} \quad (13)$$

Thus, according to Definition 1 the covariance function $\text{Cov}_X(t_1, t_2)$ is continuous symmetric non-negative definite. Consequently, instead of calculating the k th component $e_k(t)$ from its definition (9), a simpler way to obtain $e_k(t)$ can be developed for a centred stochastic field given by its covariance function $\text{Cov}_X(t_1, t_2)$.

Theorem 2. K-L expansion (Loève, 1977; Ash, 1990). A centred continuous real-valued stochastic process $X(t) \ t \in [a, b]$ with covariance function $\text{Cov}_X(t_1, t_2)$ $t_1, t_2 \in [a, b]$ can be expanded as follows:

$$X(t) = \sum_{i=1}^{\infty} z_i e_i(t) \quad (14)$$

such that:

$$E(z_i z_j) = \delta_{ij} \lambda_i \quad (15)$$

where δ_{ij} denotes the Kronecker δ ; $\{z_i\}_{i \in \mathbb{N}}$ are a sequence of mutually uncorrelated real-valued random variables; and $\{\lambda_i\}_{i \in \mathbb{N}}$ and $\{e_i(t)\}_{i \in \mathbb{N}}$ are the positive eigenvalues and the corresponding orthonormal eigen-functions generated by the given covariance function $\text{Cov}_X(t_1, t_2)$ according to Mercer's theorem. Owing to the following relationship:

$$E \left(\int_a^b \left(\sum_{i=1}^M z_i e_i(t) \right)^2 dt \right) = \sum_{i=1}^M \lambda_i \quad (16)$$

the convergence of series (14) is in the quadratic mean (Loève, 1977) and is absolute and uniform. In particular, if the process $X(t) \ t \in [a, b]$ is Gaussian, then $\{z_i\}_{i \in \mathbb{N}}$ are stochastically independent Gaussian random variables and the convergence of equation (14) becomes almost surely absolute and uniform (Loève, 1977).

Since, series (14) is of absolute convergence, the sequence of $z_i e_i(t)$ appearing in equation (14) is arbitrary, and for the most efficient approximation, it can be chosen as the descending order of the corresponding eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$. The number of terms needed in the approximate expansion, i.e. the partial sum of infinite series (14), is determined by the trace relation (6) and the accuracy is controlled by equation (16). For example, if the trace of a given covariance function is $\text{Tr}(\text{Cov}_X(t_1, t_2)) = 10$ and the associated eigenvalues are obtained as $\{\lambda_1, \lambda_2, \lambda_3, \dots\} = \{4, 3, 1.5, \dots\}$, then an approximation of 70 per cent accuracy can be readily achieved by truncating series (14) from the second term, i.e. $X(t) \approx z_1 e_1(t) + z_2 e_2(t)$ where $E(z_1^2) = \lambda_1 = 4$, $E(z_2^2) = \lambda_2 = 3$, $e_1(t)$ and $e_2(t)$ are the corresponding eigen-functions. This is because:

$$\frac{\lambda_1 + \lambda_2}{\text{Tr}(\text{Cov}_X(t_1, t_2))} = \frac{4+3}{10} = 70 \text{ per cent.}$$

2.3 Generalizations and formulations

The first generalization of the aforementioned Definition 1, Theorems 1 and 2 involves replacing the one-dimensional interval $[a, b]$ with any compact Hausdorff space C and Lebesgue measure on $[a, b]$ by a finite countable additive measure μ on the

Borel algebra of S whose support is C (Loève, 1977; Zaanen, 1964; Courant and Hilbert, 1953; Simon, 1979; Ash, 1990). The second generalization deals with measurable functions instead of only continuous functions. Then essentially the two theorems still hold under these generalizations. Consequently, as will be demonstrated in the following paragraphs, stochastic field discretization based on the K-L expansion can be applied, with mathematical rigour, to various covariance functions and domains of any dimension.

Given the expectation function $m(\tau)$ and the covariance function $\text{Cov}_X(\tau_1, \tau_2)$ of a continuous real stochastic field $X(\tau)$ in a multidimensional domain D , i.e. $\tau, \tau_1, \tau_2 \in D$, the stochastic field $X(\tau)$ can be decomposed into:

$$X(\tau) = m(\tau) + X_s(\tau) \quad (17)$$

where $X_s(\tau)$ is a centred continuous stochastic field with the same covariance function $\text{Cov}_X(\tau_1, \tau_2)$. According to the generalized K-L expansion the centred stochastic field $X_s(\tau)$ $\tau \in D$ has the following expansion:

$$X_s(\tau) = \sum_{i=1}^{\infty} z_i e_i(\tau) \quad (18)$$

where $\{z_i\}_{i \in \mathbb{N}}$ is a sequence of uncorrelated random variables as shown in expression (15); $\{\lambda_i\}_{i \in \mathbb{N}}$ and $\{e_i(\tau)\}_{i \in \mathbb{N}}$ are the eigenvalues and eigen-functions of the following characteristic equation:

$$\int_D \text{Cov}_X(\tau_1, \tau_2) e(\tau_2) d\tau_2 = \lambda e(\tau_1) \quad (19)$$

where $d\tau_2$ denotes the infinitesimal element in D . Moreover, according to the generalized Mercer theorem, the covariance function $\text{Cov}_X(\tau_1, \tau_2)$ can be expressed as:

$$\text{Cov}_X(\tau_1, \tau_2) = \sum_{i=1}^{\infty} \lambda_i e_i(\tau_1) e_i(\tau_2). \quad (20)$$

In particular:

$$\text{Tr}(\text{Cov}_X(\tau_1, \tau_2)) \triangleq \int_D \text{Cov}_X(\tau, \tau) d\tau = \sum_{i=1}^{\infty} \lambda_i \quad (21)$$

Hence, substituting equation (18) into equation (17) yields:

$$X(\tau) = m(\tau) + \sum_{i=1}^{\infty} z_i e_i(\tau) \quad (22)$$

that is the continuous stochastic field $X(\tau)$ can be represented in terms of a series of mutually uncorrelated random variables $\{z_i\}_{i \in \mathbb{N}}$ and deterministic orthonormal functions $\{e_i(\tau)\}_{i \in \mathbb{N}}$. Naturally, the most efficient approximation is achieved by choosing the sequence as the descending order of the corresponding eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$ and truncating the infinite series in equation (22) up to a term determined by the trace relation (21) and the required accuracy.

A key issue in the K-L expansion is to solve the positive eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$ and orthonormal eigen-functions $\{e_i(\tau)\}_{i \in \mathbb{N}}$ from the characteristic equation (14), which is a homogeneous Fredholm integral equation of the second kind. This is discussed in Section 4 in which a semi-analytic solution is obtained.

3. Multiple Fourier series

A complete exploration of multiple Fourier series is not only beyond the scope of this paper but also unnecessary, therefore, the following discussion only covers preliminary knowledge required in Section 4.

For a pair of real functions $f(x), g(x) \in L^2(\mathbf{T}^n)$ $x \in \mathbf{T}^n$ where \mathbf{T}^n denotes a hyper-cuboid in an n -dimensional real space \mathbb{R}^n :

$$\mathbf{T}^n \triangleq \{x|x = (x_1, \dots, x_n) \in \mathbb{R}^n, -a_i \leq x_i \leq a_i, a_i > 0, i = 1, \dots, n\}, \quad (23)$$

introduce an operator $\langle \bullet, \bullet \rangle$ which is defined as follows:

$$\langle f(x), g(x) \rangle \triangleq \int_{\mathbf{T}^n} f(x)g(x)dx \quad (24)$$

It is trivial to verify that the above operator satisfies all the requirements for a general inner product, and the norm induced by this inner product is given by:

$$\|f(x)\| \triangleq \langle f(x), f(x) \rangle^{\frac{1}{2}} \quad (25)$$

Theorem 3. For multiple Fourier series defined on \mathbf{T}^n :

$$\{\phi_i(x)\}_{i \in \mathbb{N}} \triangleq \frac{1}{\sqrt{2^{n-1} \prod_{j=1}^n a_j}} \left\{ \frac{1}{\sqrt{2}}, \cos \sum_{j=1}^n m_j \omega_j x_j, \sin \sum_{j=1}^n m_j \omega_j x_j \right\} \quad m = (m_1, \dots, m_n) \neq 0 \\ m_1, \dots, m_n \in \{0, 1, 2, \dots\} \quad (26)$$

where $\omega_j = \pi/a_j$, the following properties hold:

(1) orthogonality, i.e.:

$$\langle \phi_i(x), \phi_j(x) \rangle = \delta_{ij} \quad (27)$$

(2) completeness, i.e. if $f(x) \in L(\mathbf{T}^n)$ and:

$$\langle f(x), \phi_i(x) \rangle = 0 \quad \forall i \in \mathbb{N} \quad (28)$$

then $f(x) = 0$ almost everywhere in \mathbf{T}^n (Körner, 1989).

For the sake of simplicity, the real form of multiple Fourier series (26), which is originally indexed by a non-zero integer vector $m = (m_1, \dots, m_n)$ where $m_1, \dots, m_n \in \{0, 1, 2, \dots\}$, is re-indexed by $i \in \mathbb{N}$ in ascending order. According to Theorem 3, the Fourier function set $\{\phi_i(x)\}_{i \in \mathbb{N}}$ forms a complete standard orthogonal function basis in the hyper-cuboid \mathbf{T}^n , and this fact will play a crucial role in the following discretization procedure of wide-sense stationary stochastic fields.

4. Fourier discretization of wide-sense stationary stochastic fields

4.1 The formulation

Without loss of generality, this paper considers a wide-sense stationary stochastic field defined in a hyper-cuboid, i.e. $X(\tau) \in \mathbf{T}^n$. For the sake of simplicity, it is assumed that $X(\tau)$ are real (scalar) random variables for all τ , but the result obtained in this work is applicable without change to vector-valued random variables. Multiple Fourier functions defined in equation (26) are taken as the function basis to format the stochastic field, and the well-posedness, convergence and stability of the solution are strictly justified by the orthogonality and completeness properties given in Theorem 3. The K-L expansion is chosen as the approximation criterion because of its readily available simple form as shown in Section 2.

The stochastic field $X(\tau)$ is first centralized so that equation (2) accurately holds and according to expansion (22), the approximate stochastic field $Y(\tau)$ takes the following form:

$$Y(\tau) = m(\tau) + \sum_{i=1}^M z_i e_i(\tau) \quad (29)$$

Note that a truncated finite series is used above. The choice of the number of terms, M , will be discussed later.

The next task is to solve the eigen-functions $e_i(\tau) i = 1, \dots, M$ of the characteristic equation (19) in terms of Fourier basis functions $\{\phi_i(\tau)\}_{i \in \mathbb{N}}$ in equation (26), or more specifically, to determine all the principal directions of the given covariance function $\text{Cov}_X(\tau_1, \tau_2)$ in a Fourier coordinate system. This problem of determining eigen-functions of the characteristic equation (19) is similar to the standard eigenvalue problem in linear algebra. The problem in linear algebra is formulated in Euclidean space equipped with a Cartesian coordinate system, while the current eigen-functions are formulated in the separable Hilbert space:

$$\mathcal{F} \triangleq \text{span}\{\phi_i(\tau)\}_{i \in \mathbb{N}} \quad (30)$$

equipped with a Fourier coordinate system. However, all other features are essentially the same.

According to the completeness of the Fourier basis functions $\{\phi_i(\tau)\}_{i \in \mathbb{N}}$, the basis functions $e_i(\tau)$ in equation (29) can be decomposed into:

$$e_i(\tau) = \sum_{j=1}^{\infty} c_{ij} \phi_j(\tau) \quad i = 1, 2, \dots, M \quad (31)$$

where c_{ij} are unknown coefficients. Substituting this expansion into the characteristic equation (19) yields:

$$\int_{\mathbf{T}^n} \left[\text{Cov}_X(\tau_1, \tau_2) \sum_{j=1}^{\infty} c_{ij} \phi_j(\tau_2) \right] d\tau_2 - \lambda \sum_{j=1}^{\infty} c_{ij} \phi_j(\tau_1) = 0 \quad i = 1, \dots, M \quad (32)$$

In a numerical treatment, only the first $N \geq M$ terms in the infinite series (31) are considered, which leads to an error field:

$$\Delta_i(\tau_1) = \int_{\mathbb{T}^n} \left[\text{Cov}_X(\tau_1, \tau_2) \sum_{j=1}^N c_{ij} \phi_j(\tau_2) \right] d\tau_2 - \lambda \sum_{j=1}^N c_{ij} \phi_j(\tau_1) \quad i = 1, \dots, M \quad (33)$$

Representation
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Naturally, the best approximation should minimize this error field over \mathbb{T}^n in a certain measure sense. Using the norm defined in equation (25), probably the most convenient measure, a simple optimization problem follows immediately:

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$$c_{ij} = \arg \min_{c_{ij}} (\|\Delta_i(\tau_1)\|) \quad i = 1, \dots, M \quad (34)$$

Letting:

$$\mathcal{F}_N \triangleq \text{span}\{\phi_1(\tau), \dots, \phi_N(\tau)\} \quad (35)$$

denote a subspace of \mathcal{F} the aim of the optimization problem (34) is to find the best approximations of $e_i(\tau)$ $i = 1, \dots, M$ in \mathcal{F}_N measured by the norm (25) of \mathcal{F} . In the theory of Hilbert space, this is equivalent to setting the error field (33) into the null space \mathcal{F}_N^\perp of \mathcal{F}_N such that:

$$\mathcal{F}_N^\perp \oplus \mathcal{F}_N = \mathcal{F} \quad (36)$$

and:

$$\mathcal{F}_N^\perp \perp \mathcal{F}_N \quad (37)$$

where \oplus denotes the direct sum between two linear spaces. Hence:

$$\langle \phi_k(\tau_1), \Delta_i(\tau_1) \rangle = 0 \quad k = 1, \dots, N \text{ and } i = 1, \dots, M \quad (38)$$

Substitution of equation (33) into equation (38) yields:

$$\begin{aligned} & \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \phi_k(\tau_1) \left[\text{Cov}_X(\tau_1, \tau_2) \sum_{j=1}^N c_{ij} \phi_j(\tau_2) \right] d\tau_2 d\tau_1 \\ &= \int_{\mathbb{T}^n} \phi_k(\tau_1) \left[\lambda \sum_{j=1}^N c_{ij} \phi_j(\tau_1) \right] d\tau_1 \quad k = 1, \dots, N \text{ and } i = 1, \dots, M \end{aligned} \quad (39)$$

To simplify the formulation, an operator $(\bullet \circ \bullet)$ acting on Fourier basis functions $v\{\phi_i(\tau)\}_{i \in \mathbb{N}}$ is defined as:

$$(\phi_{i_1} \circ \phi_{i_2}) \triangleq \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \text{Cov}_X(\tau_1, \tau_2) \phi_{i_1}(\tau_1) \phi_{i_2}(\tau_2) d\tau_1 d\tau_2 \quad \forall \phi_{i_1}, \phi_{i_2} \in \{\phi_i(\tau)\}_{i \in \mathbb{N}} \quad (40)$$

From the symmetric non-negative definite property of a covariance function, it is easy to verify that the operator $(\bullet \circ \bullet)$ is symmetric non-negative definite, i.e.:

$$(i) \quad (\phi_{i_1} \circ \phi_{i_2}) = (\phi_{i_2} \circ \phi_{i_1}) \quad \forall \phi_{i_1}, \phi_{i_2} \in \{\phi_i(\tau)\}_{i \in \mathbb{N}} \quad (41a)$$

$$(ii) \quad (\phi_{i_1} \circ \phi_{i_1}) \geq 0 \quad \forall \phi_{i_1} \in \{\phi_i(\tau)\}_{i \in \mathbb{N}} \quad (41b)$$

As $\{\phi_i(\tau)\}_{i \in \mathbb{N}}$ are a mutually orthonormal function set, the system of equation (39) becomes:

$$\begin{bmatrix} (\phi_1 \circ \phi_1) & (\phi_1 \circ \phi_2) & \cdots & \cdots & (\phi_1 \circ \phi_N) \\ (\phi_2 \circ \phi_1) & (\phi_2 \circ \phi_2) & \cdots & \cdots & (\phi_2 \circ \phi_N) \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ (\phi_N \circ \phi_1) & (\phi_N \circ \phi_2) & \cdots & \cdots & (\phi_N \circ \phi_N) \end{bmatrix} \begin{bmatrix} c_{i_1} \\ c_{i_2} \\ \vdots \\ \vdots \\ c_{i_N} \end{bmatrix} = \lambda_i \begin{bmatrix} c_{i_1} \\ c_{i_2} \\ \vdots \\ \vdots \\ c_{i_N} \end{bmatrix} \quad i = 1, \dots, M \quad (42a)$$

or in matrix form:

$$\boldsymbol{\varphi} \mathbf{c}_i = \lambda_i \mathbf{c}_i \quad i = 1, \dots, M \quad (42b)$$

From the symmetric non-negative definite property of $\text{Cov}_X(\tau_1, \tau_2)$, it is trivial to prove that the $N \times N$ constant matrix $\boldsymbol{\varphi}$ is symmetric and non-negative definite. Furthermore, due to the harmonic essence of wide-sense stationary stochastic fields, as well as the orthogonality and completeness of Fourier basis functions, matrix $\boldsymbol{\varphi}$ is diagonally dominant.

Equation (42) is a standard algebraic eigenvalue problem. Moreover, due to the diagonally dominant property of $\boldsymbol{\varphi}$ and the well-known Gershgorin circle theorem (Brualdi and Mellendorf, 1994), the diagonal entries of matrix $\boldsymbol{\varphi}$ provide a very good estimate to the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$, which in turn significantly reduces the computational cost of solving this algebraic eigenvalue problem. As shown in equation (31), the orthonormal eigen-functions $e_i(\tau)$ are determined by those normalized eigenvectors \mathbf{c}_i corresponding to the first M largest eigenvalues λ_i , $i = 1, \dots, M$, from which the stochastic field discretization (29) is finally obtained. The explicit stochastic field representation (29) obtained from the proposed method provides a semi-analytic solution to the K-L expansion.

4.2 Computational issues

Computational costs involved in the proposed method include two parts:

- (1) the cost of constructing the $N \times N$ symmetric non-negative definite matrix $\boldsymbol{\varphi}$; and
- (2) the cost associated with the solution of the first M largest eigenvalues and their corresponding eigenvectors in equation (42).

Nevertheless, each entry of matrix $\boldsymbol{\varphi}$, i.e. $(\phi_i \circ \phi_j)$, involves calculating a generalized double-integration of oscillation functions as shown in equation (40), which can be computationally expensive for those entries with a small wave length corresponding to large nominal wave numbers i and j . Consequently, this computational issue is discussed in detail below.

As Fourier basis functions $\{\phi_i(\tau)\}_{i \in \mathbb{N}}$ are normalized (therefore, bounded by 1 in norm (25)) and periodic over \mathbf{T}^n , the value of the generalized double-integration $(\phi_{i_1} \circ \phi_{i_2})$ in equation (40) is dominated by the weighted function $\text{Cov}_X(\tau_1, \tau_2)$ to a large extent. It is further noticed that in practice, the covariance function $\text{Cov}_X(\tau_1, \tau_2)$ is always single-peaked in $\mathbf{T}^n \times \mathbf{T}^n$ (Figure 2) and tends to be zero over the most part of the definition domain. Let $\mathbf{T}_{R^+}^{2n} \subset \mathbf{T}^n \times \mathbf{T}^n$ denote the effective sub-domain in which

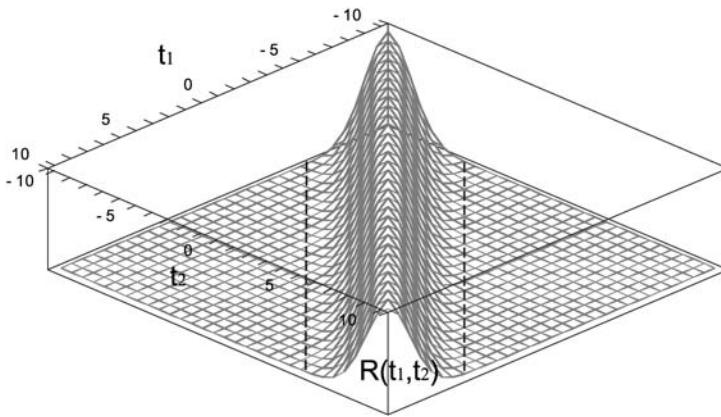


Figure 2.
Single-peaked covariance
function of a
one-dimensional
stochastic process

$\text{Cov}_X(\tau_1, \tau_2)$ is relatively large; hence the generalized double-integration ($\phi_i \circ \phi_j$) only needs to be calculated within $\mathbf{T}_{R^+}^{2n}$ which is usually a rather small area compared to the whole definition domain $\mathbf{T}^n \times \mathbf{T}^n$.

An example of a one-dimensional stochastic process is shown in Figure 2, for which the double-integration can be limited to the narrow band between the two dashed lines. Both compound Gaussian integration and compound Simpson's integration are implemented for the calculation and the number of integration points is adaptively determined by the corresponding nominal wave numbers i and j to ensure that, on every independent dimension of $\phi_i(\tau)$ and $\phi_j(\tau)$, at least five integration points are distributed in a single wave length. Numerical experiments have shown that these two simple adaptive strategies accelerate the process of constructing matrix φ up to ten times faster, depending on the covariance function and the definition domain concerned. Rather surprisingly, it is found that, for the generalized double-integration considered here, the overall performance of the Gaussian integration and Simpson's integration are essentially the same, although there is a visible advantage of the Gaussian scheme over Simpson's in terms of accuracy.

4.3 Discussions

The new representation method for stochastic fields presented above possesses several distinguishing features, which are highlighted below:

- In the above formulation, the stochastic field $X(\tau)$ is assumed as having definition in the entire domain of hyper-cuboid \mathbf{T}^n , over which Fourier basis functions $\{\phi_i(\tau)\}_{i \in \mathbb{N}}$ are mutually orthogonal. This assumption is necessary for the success of simplifying equation (39) into a standard algebraic eigenvalue problem (42). However, this is not a restriction for practical problems, since no matter what shape the real structure is, the stochastic field affixed to it can always be seamlessly expanded into a virtual hyper-cuboid. As a matter of fact, for a wide-sense stationary stochastic field, its definition domain is totally unrelated to the underlying physical geometry. Consequently, the solution provided by this method is independent of the actual structural shape, which is a distinct advantage over other methods whose computational efficiency and results vary with the geometric configuration of the structure concerned.

- In the current formulation, the first M eigen-functions $e_i(\tau)$ used in the K-L expansion (29) are expressed as the linear combination of N multiple Fourier functions $\phi_j(\tau)$. Clearly, N must be larger than M to ensure the linear independence of the computed eigen-functions. Moreover, N has to be sufficiently large to accurately describe the spectra of the stochastic field $X(\tau)$. Indeed, for wide-sense stationary stochastic fields, N can be explicitly determined by the dimension of \mathbf{T}^n and the spectrum of $\text{Cov}_X(\tau_1, \tau_2)$.
- When a finite element mesh is used to discretize stochastic fields, the size of each element must be sufficiently small. Specifically, the element size ought to be examined in comparison to the scale of correlation associated with the stochastic field under consideration. In addition, the element size must also be addressed from the stress/strain gradient point of view; a finer mesh is needed in a location where the gradient is more significant. The finite element method approximates the solution by virtue of assumed shape functions that are not necessarily, and in fact most probably not, consistent with the spatial variability of the stochastic field, thus the size of the finite element discretization must satisfy the requirements from both the stress gradient consideration and the stochastic field variability. This issue is not easily resolved, however, even when a separate background mesh is generated for the stochastic field (Yamazaki *et al.*, 1988; Harada and Shinozuka, 1986). In this respect, the proposed method provides a distinct answer, i.e. a semi-analytic solution to the K-L expansion.
- As addressed earlier, the standard finite element method has been used in the literature (Ghanem and Spanos, 1991; Babuška and Chatzipantelidis, 2002; Frauenfelder *et al.*, 2005) to solve the characteristic equation (19), and an error-minimizing equation, similar to equation (38) in form, is achieved via a Galerkin approach. Then, the final discrete equation is transformed into a generalized algebraic eigenvalue problem, i.e.:

$$\mathbf{Ac} = \lambda \mathbf{Bc} \quad (43)$$

where \mathbf{A} is a $q \times q$ symmetric non-negative definite full matrix subject to both the covariance function and the FE-mesh structure; \mathbf{B} a $q \times q$ symmetric positive definite sparse matrix exclusively determined by the mesh structure; \mathbf{c} a q dimensional eigenvector; λ the corresponding eigenvalue; and q is equal to the total number of nodes in the FE-mesh utilized in discretization. For the reason explained in the last item, q becomes increasingly large as the correlation distance of the stochastic field under study decreases, and in 3D cases it can easily grow up to 10^6 which will reach the memory limit of current normal computers (Frauenfelder *et al.*, 2005). However, the final discrete equation of the proposed method is a standard algebraic eigenvalue problem (42) with respect to a diagonally dominant matrix φ determined by the covariance function and multiple Fourier series. The diagonally dominant property of matrix φ makes the solution of its eigen-structure much easier. Furthermore, the order of matrix φ , N , is not at the same level as those of matrices \mathbf{A} and \mathbf{B} . In fact, $N \ll q$ for wide-sense stationary stochastic fields. These significant differences between φ , and \mathbf{A} and \mathbf{B} arises from the fact that wide-sense stationary stochastic fields have a harmonic essence.

5. Numerical verifications

To demonstrate the performance of the proposed method, two types of stochastic fields described, respectively, by the exponential covariance function and the Gaussian covariance function are examined in detail. Without loss of generality, all the stochastic fields considered in the following examples are assumed as having mean zero, i.e. $m(\tau) \equiv 0$.

5.1 Example 1

Consider a set of stochastic fields $X(t)$ $t \in [-a, b]$ described by covariance function $e^{-|t_1 - t_2|/b}$, where the correlation distance is measured by $b > 0$. The aim of this one-dimensional example is to show that the proposed method can efficiently and accurately capture the eigenvalues λ_i as well as the eigen-functions $e_i(t)$ required by the K-L expansion.

For this simple case, the following analytic solution is available for the corresponding characteristic equation (4) (Ghanem and Spanos, 1991; Vantrees, 1968):

$$\lambda_i = \frac{2b}{1 + b^2 \omega_i^2} \quad i = 1, 2, 3, \dots \quad (44)$$

$$e_i(t) = \begin{cases} \frac{\cos(\omega_i t)}{\sqrt{a + \frac{\sin(2\omega_i a)}{2\omega_i}}} & i = 1, 3, 5, \dots \\ \frac{\sin(\omega_i t)}{\sqrt{a - \frac{\sin(2\omega_i a)}{2\omega_i}}} & i = 2, 4, 6, \dots \end{cases} \quad (45)$$

where $\omega_1 \leq \omega_2 \leq \omega_3 \leq \dots$ are given by the following transcendental equations:

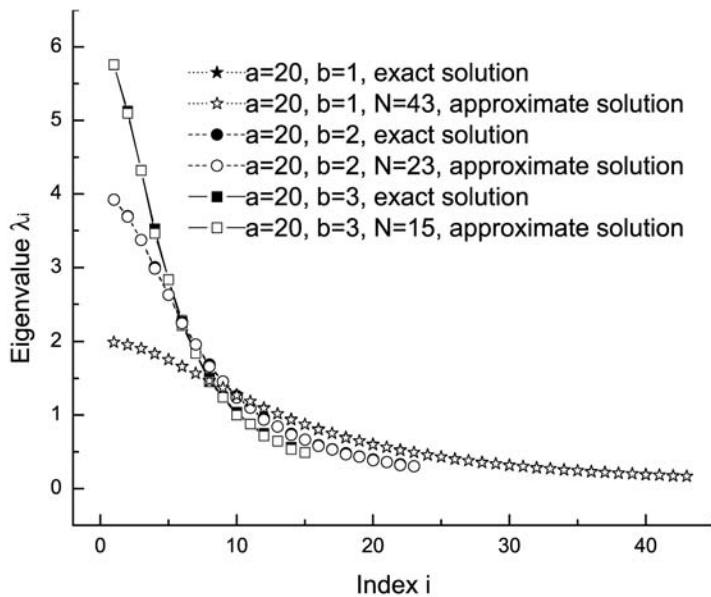
$$1 - b\omega \tan(\omega a) = 0 \quad (46a)$$

or:

$$b\omega + \tan(\omega a) = 0 \quad (46b)$$

In Figure 3, the approximate eigenvalues calculated from the proposed method are compared with the exact values determined by equation (44). Different numbers of Fourier terms are used in the numerical approximations, and the larger the correlation distance, the fewer the terms that are needed. Good agreements are observed in all the cases with different correlation distances. Figure 4 shows the comparison between the exact eigen-functions determined by equation (45) and the approximate ones obtained by the proposed method. Unlike the FE-mesh-based solution which could not capture eigen-functions very effectively (Ghanem and Spanos, 1991), the proposed method also demonstrates very good agreements on eigen-functions with only a small number of Fourier terms.

Note: the exponential covariance function used in this example is SOLELY employed to verify the effectiveness and accuracy of the proposed method in solving the characteristic equation. Although the exponential covariance of the form $e^{-|\tau|/b}$ has been widely used in the SFEM literature (Liu *et al.*, 1986a, b; Kami ski and Kleiber, 1996; Hien and Kleiber, 1997; Ghanem and Spanos, 1991; Babuška and

**Figure 3.**

Eigenvalues associated with a set of exponential covariance functions

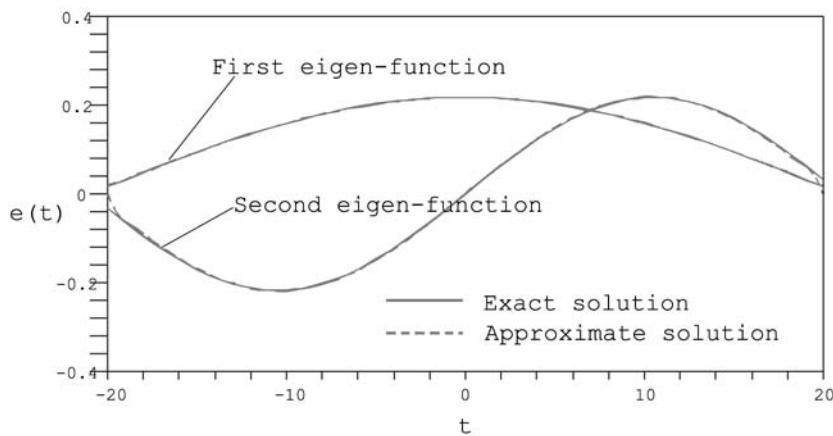
Chatzipantelidis, 2002; Frauenfelder *et al.*, 2005; Vanmarcke and Grigoriu, 1983; Li and Derkiureghian, 1993), the authors do not advocate its usage in describing material properties of random media as its corresponding stochastic fields are non-differentiable almost everywhere (Yaglom, 2004).

5.2 Example 2

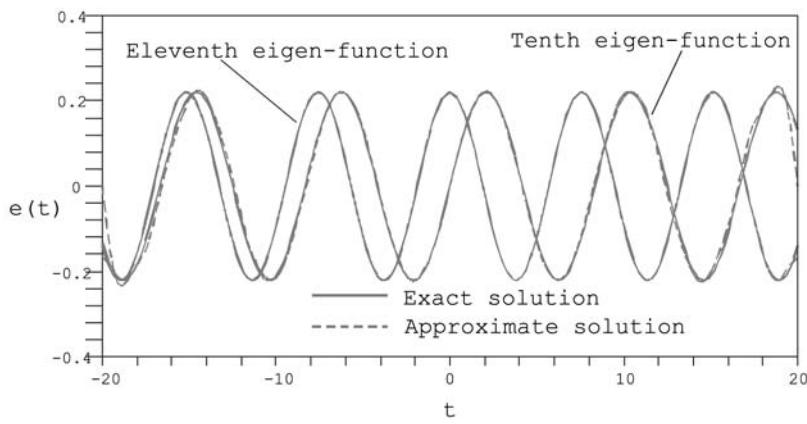
Consider a set of stochastic fields $X(t)$ $t \in [-a, a]$ described by covariance function $e^{-((t_1-t_2)^2/\sigma^2)}$, where the correlation distance is indicated by σ . The aim of this example is to demonstrate the efficiency of the proposed method in discretizing stochastic fields.

As shown in Figure 5, the diagonal entries of the diagonally dominant matrix φ give a very good estimate to the corresponding eigenvalues. Therefore, it is trivial to solve the standard algebraic eigenvalue problem (42) and obtain the stochastic field discretization.

In a SFEM treatment, the computational cost always depends on the total number of random variables in the system. Therefore, a good discretization method must approximate the stochastic field with as few random variables as possible. In the K-L expansion, this number M is determined by the accuracy required and the trace relation (6) due to the quadratic mean convergence (16). In Figure 6, the x -axis denotes M , the y -axis denotes the partial sum of M eigenvalues and the dashed lines mark the theoretical trace values calculated from equation (6). As expected, the number of random variables required to achieve good approximations are reasonably small. It is also the case for the number of Fourier terms used. The larger the correlation distance, the smaller the value of M , Figure 7, in which stochastic fields are represented by their corresponding covariance functions, illustrates the difference between the original stochastic field and the approximate one obtained by the proposed method.



(a) The first and second eigen-functions



(b) The tenth and eleventh eigen-functions

Figure 4.
Eigen-functions associated
with an exponential
covariance function
 $(a = 20, b = 1)$

5.3 Example 3

A 2D stochastic field $X(s,t)$ with covariance function $e^{-((s_1-s_2)^2+((t_1-t_2)^2/4^2))}$ where $(s,t) \in [-12, 12] \times [-14, 14]$ is considered.

Although this problem can be solved by using 2D Fourier series, as the two-variable covariance function is separable, a more efficient way is to first solve two separate one-dimensional fields, i.e. $e^{-(s_1-s_2)^2}$ with $s_1, s_2 \in [-12, 12]$ and $e^{-(t_1-t_2)^2/4}$ with $t_1, t_2 \in [-14, 14]$, and then multiply them together. This is a standard technique (Ghanem and Spanos, 1991; Frauenfelder *et al.*, 2005) but sometimes the technique cannot be applied to a FE-mesh-based discretization because its domain is not always a hyper-cuboid.

The eigenvalues and eigen-functions are shown in Figures 8 and 9, respectively: the eigen-functions are unsymmetrical due to the fact that the stochastic field is not

Figure 5.
Eigenvalues and diagonal entries of matrix φ
(associated with different Gaussian covariance functions)

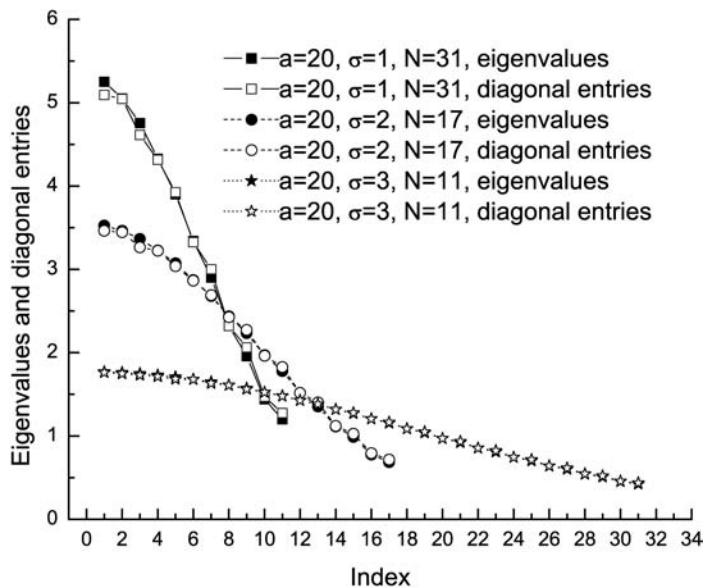
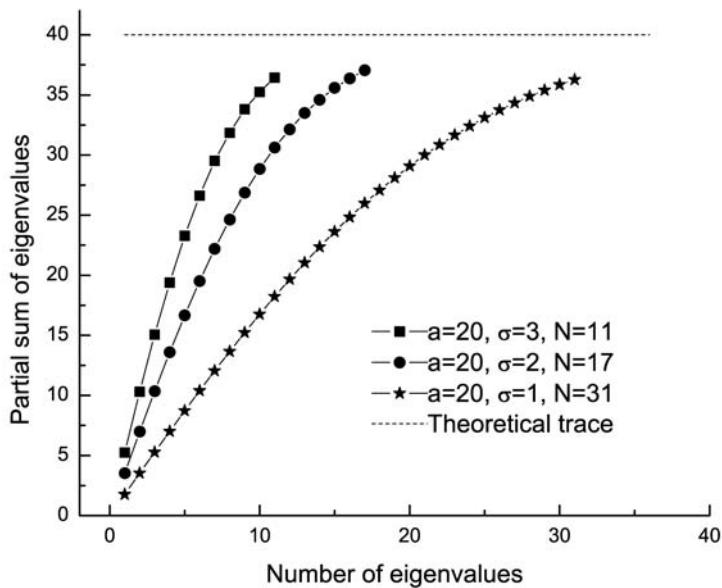
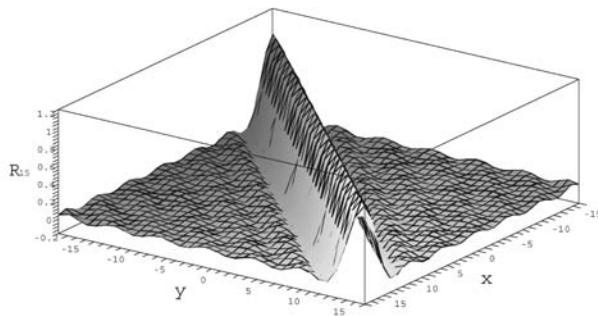
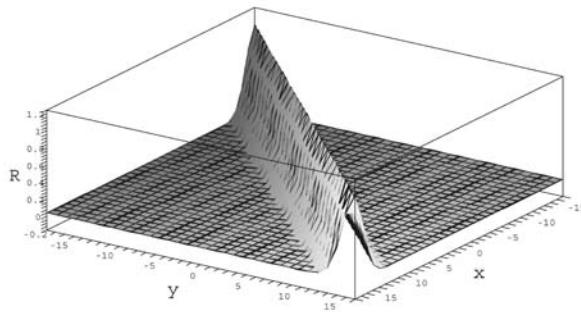


Figure 6.
Eigenvalue approximations associated with different Gaussian covariance functions

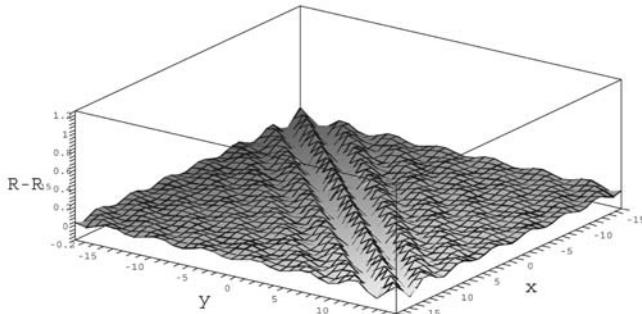




(a) Covariance function of the approximate stochastic field with $N = 15$



(b) Covariance function of the exact stochastic field



(c) Difference between the approximate and exact stochastic field

Figure 7.
Comparison between the exact and approximate stochastic fields (defined by a Gaussian covariance function with $a = 20$, $\sigma = 2$)

symmetric with regard to coordinates s and t . The stochastic field discretization can then be achieved by substituting the obtained eigenvalues and eigen-functions into equation (29).

6. Conclusions

A new method for the representation of wide-sense stationary stochastic fields is proposed, which chooses multiple Fourier functions as the function basis to format the

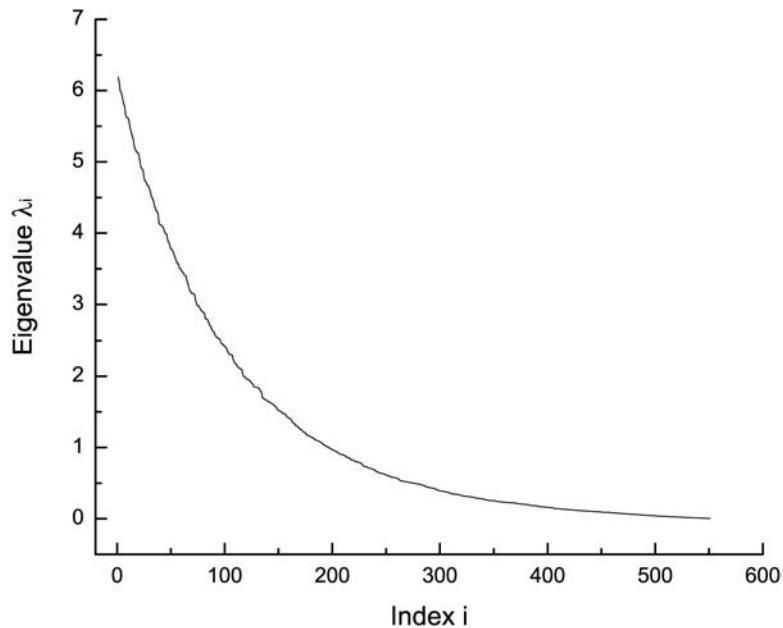
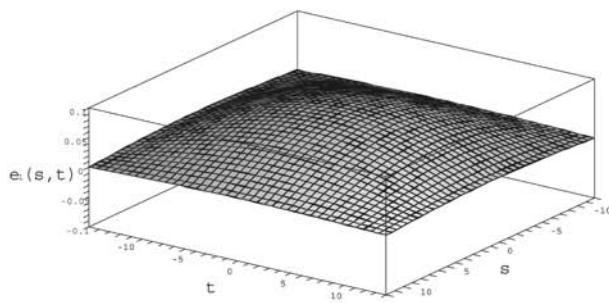


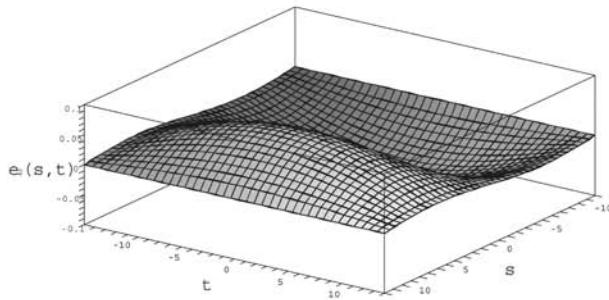
Figure 8.
Eigenvalues of a 2D
stochastic field with
Gaussian covariance
function

domain and the K-L expansion as the criterion to approximate the stochastic field given by its expectation and covariance functions. The main features of the method are summarised as follows:

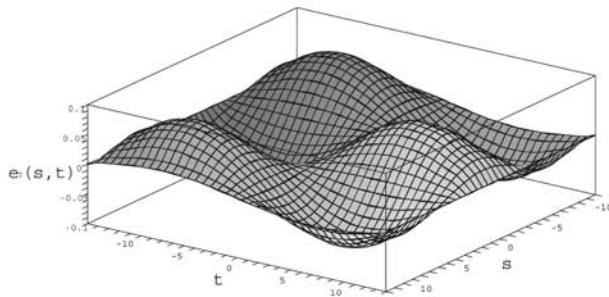
- As a semi-analytic solution to the K-L expansion, the new method retains all the original properties of the expansion.
- It is efficient with regard to both the number of random variables and the number of Fourier terms required to achieve a given level of accuracy.
- As multiple Fourier functions are taken as the basic building blocks, the overall performance of the proposed method is not sensitive to the correlation distance or the dimension of the stochastic field, especially for wide-sense stationary stochastic fields whose probabilistic characteristics are invariant over the domain.
- The new method expands stochastic fields in hyper-cuboids which are independent of the real structural shape. Moreover, a dimensional reduction technique can always be expected to further save computational costs as long as the covariance function concerned is separable.
- The computational cost of the new method is very competitive, not only because its final discrete equation is a standard algebraic eigenvalue problem with a diagonally dominant matrix, instead of a generalized eigenvalue problem from FE-mesh-based solution results, but also because the order of this final matrix is much smaller than that in a FE-mesh-based solution.



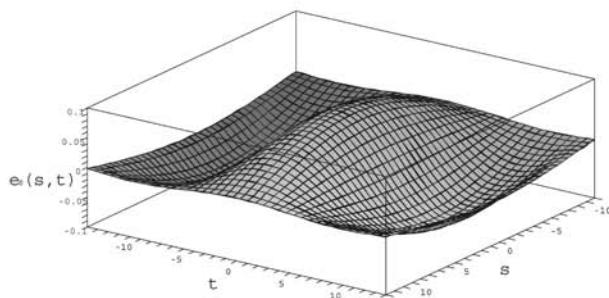
(a) The first eigen-function



(b) The second eigen-function



(c) The seventh eigen-function



(d) The eighth eigen-function

Figure 9.
Eigen-functions of a 2D
stochastic field with
Gaussian covariance
function

- In terms of a vector of mutually uncorrelated random variables and multiple Fourier series, the presented method provides an explicit representation for a stochastic field in a closed form. Therefore, it can easily be integrated into a large family of analytical tools and numerical algorithms for problems with a stochastic nature.

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