A quasi-static crack propagation simulation based on shape-free hybrid stress-function finite elements with simple remeshing

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Abstract

In this paper, a new shape-free multi-node singular hybrid stress-function (HSF) element and a shape-free 8-node plane HSF element proposed recently are employed to simulate the quasi-static 2D crack propagation problem. Compared with other well-known methods, such new scheme exhibits four advantages: (i) for the singular element, the shape and the number of nodes can be flexibly adjusted as required; (ii) high precision for stress intensity factors (SIF) can be obtained due to the advantages of the HSF method; (iii) only simple remeshing with a very coarse mesh is needed for each simulation step; (iv) unstructured mesh containing extremely distorted elements can be used without losing precision. It demonstrates that the proposed scheme is an effective technique for dealing with crack propagation problems.

Keywords: Finite element; Hybrid stress-function (HSF) element; Shape-free; Crack propagation; Simple remeshing

1. Introduction

The finite element method is usually considered as an effective tool for simulating the behaviors of materials or structures in fracture mechanics problems. Over the past few decades, some schemes have been successfully established in numerous literatures.

The main characteristic of the crack problem is that, the stresses at the crack tip are singular. In order to catch such singularity, the element models around the crack tip need special treatments. In the most usual
strategy for plane crack problem, the 8-node quarter-point elements or the 6-node quarter-point elements (collapsible quadrilateral) are often used since they can produce the singular stress fields at the vicinity of the crack tip [1–4]. These singular element models must be located at the crack tip and linked with the conventional isoparametric elements located in other domain. However, some drawbacks were found existing in such method: (i) The mesh density around the crack tip must be quite high, otherwise, the accuracy of the resulting stress intensity factors (SIF) cannot be guaranteed. (ii) When the 8-node quarter-point elements are used, different crack propagation directions may be obtained if the meshes are different. (iii) When the 6-node quarter-point elements (collapsed quadrilateral) are used, in order to produce acceptable results, a structured mesh near the crack tip that looks like spider’s web (as showed in Fig. 1) is needed, which is inconvenient to be remeshed after the crack extends. (iv) Due to the inherent defects existing in usual finite element method, once unstructured meshes or too many severely distorted elements are used, the precision may be very poor.

To avoid the inconvenience caused by remeshing and direction dependency, Belytschko et al. [5–7] proposed an extended finite element method (X-FEM) for discontinuous field by adding local enrichment functions to the FEM. The X-FEM has become one of the recent research focuses in computational fracture mechanics and attracted many researchers. Its most outstanding advantage is that, it allows the crack to pass through an element without remeshing. To realize this point, a so-called level set method must be employed to describe and update the information of the crack. And the discontinue element containing crack may bring difficulty for integration [8,9]. Furthermore, since most elements used in X-FEM are displacement-based models, the accuracy of the stress solutions is relatively low. So, in order to obtain satisfactory results, the mesh density in this method is usually high. Recently, many researchers are still looking for effective ways to improve and generalize the X-FEM, such as the new meshing and the integration schemes proposed by Richardson et al. [10], the polygonal extended voronoi cell finite element model proposed by Li and Ghosh [11], the strain smoothing X-FEM proposed by Bordas et al. [12], the node-based smoothed X-FEM proposed by Vu-Bac et al. [13], the polytope finite element method proposed by Zamani and Eslami [14], the stress-hybrid quadrilateral X-FEM proposed by Dujc et al. [15], and so on.

Actually, if the finite element method used in simulating crack propagation does not need high quality and refined mesh, the scheme of remeshing is still acceptable since there is no complicated mathematical treatment existing in the simulation process, such as the scaled boundary finite element method proposed by Ooi and Yang [16,17], the polytope scaled boundary finite elements proposed by Ooi et al. [18]. From 2011 to 2013, Nourbakhshnia and Liu [19], Nguyen-Xuan et al. [20,21] successfully established a singular edge-based smoothed finite element method (ES-FEM) to simulate the quasi-static crack growth. Based on a basic mesh of linear triangular elements that can be generated automatically for problem with complicated geometries, they developed 5- and 7-node singular edge-based smoothed triangular elements at the crack tip and employed 3-node edge-based smoothed elements in other domain. Although simple remeshing is still needed, this technique can produce more accurate results than those obtained by X-FEM. However, since the low-order triangular elements are employed, the mesh density is still quite high along the path of the crack propagation.

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Fig. 1. Typical mesh with 6-node quarter-point elements at crack tip.
Some techniques only using very coarse meshes for fracture problems have also been proposed by other researchers, such as the rectangular super hybrid-element approach [22], the hybrid-Trefftz equilibrium model [23], and so on. But few of them can easily deal with the crack growth problems with complicated propagation paths. In 1980s, Long et al. [24–28] developed a kind of sub-region mixed element (SRME) method for solving 2D and 3D crack or notch problems. In this SRME method, a coupling mesh composed of displacement-based and stress-based elements is used. For example, when a fracture problem shown in Fig. 2 is being analyzed, the coupling mesh plotted in Fig. 3 is employed. One multi-node singular stress element is collocated on the stress concentration region near the tip of the crack, while the conventional 8-node displacement-based isoparametric elements are used on the even stress region at the periphery. Thus, the advantages from different element types can be fully exhibited in one problem, that is, the accuracy is improved while the computation costs decrease. However, once the SRME method is used for simulating crack propagation, it is difficult to keep a high quality mesh after remeshing, that is, some conventional isoparametric elements will be severely distorted, which leads to the loss of accuracy.

Above problem caused by the mesh distortions can be avoided. The conventional 8-node isoparametric elements, which are sensitive to mesh distortions, can be replaced by ‘shape-free’ hybrid stress-function (HSF) elements recently proposed by Cen et al. [29–33]. These plane continuum elements are extensions of the traditional hybrid-stress element [34], and immune to various severe mesh distortions. So, good results can still be obtained using unstructured meshes. Furthermore, the singular element at the crack tip can also be replaced by singular HSF element, so that only one type of element is used during the whole simulation. In this paper, the 8-node shape-free plane HSF element and a new multi-node singular HSF element will be firstly employed to simulate the 2D crack propagation problems. Compared with other methods described above, such new scheme exhibits four advantages: (i) the shape and the number of element nodes can be flexibly adjusted as required; (ii) high precision for stress intensity factors (SIF) can be obtained due to the advantages of the HSF method; (iii) only simple remeshing with a very coarse mesh is needed for each simulation step; (iv) unstructured mesh containing extremely distorted elements can be used without losing precision. It demonstrates that the proposed scheme is an effective technique for dealing with crack propagation problem.

2. The formulations of the shape-free 2D HSF elements

2.1. General formulations [29–33]

Based on the hybrid stress element method, the final finite element equation can be written as

\[
\sum_{e} K^e q^e = \sum_{e} p^e, \tag{1}
\]

where \(q^e\) is the element nodal displacement vector; \(p^e\) is the element equivalent load vector; \(K^e\) is the element stiffness matrix of the hybrid-stress function (HSF) elements for 2D problems [29–33]:

\[
K^e = H^T M^{-1} M, \tag{2}
\]

in which \(M\) is the flexibility matrix; \(H\) is the leverage matrix;

\[
M = \int_{\Gamma} S^T C S dA, \quad H = \int_{\Gamma} S^T L^T N dS, \tag{3}
\]
where $t$ is the element thickness; $A^e$, the element area; $I^e$, the element boundary. $C$ is the elastic compliance matrix, and can be expressed by

$$C = \frac{1}{E} \begin{bmatrix} 1 & -\mu' & 0 \\ -\mu' & 1 & 0 \\ 0 & 0 & 2(1 + \mu') \end{bmatrix},$$

(4)

plane stress: $E' = E'$, $\mu' = \mu$,

plane strain: $E' = E/(1 - \mu^2)$, $\mu' = \mu/(1 - \mu)$,

$E$, Young’s modulus; $\mu$, Poisson’s ratio for isotropic cases, and its form for anisotropic cases can be found in Ref. [32]. For the stresses expressed in Cartesian coordinate system, $L$ is the matrix of the direction cosines for element boundaries:

$$L = \begin{bmatrix} l & 0 & m \\ 0 & m & l \end{bmatrix},$$

(5)

where $l$ and $m$ are the direction cosines of the outer normal $n$ of the element boundaries.

Let $\mathbf{u}_r = \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix}$ be the displacement vector along element boundaries, which can be interpolated by the element nodal displacement vector $\mathbf{q}^e$:

$$\mathbf{u}_r = \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \mathbf{N} \mathbf{q}^e,$$

(6)

where matrix $\mathbf{N}$ is the interpolation function matrix for element boundary displacements, and used by Eq. (3).

The element stress fields are assumed as follows:

$$\mathbf{\sigma} = \mathbf{\sigma}^0 + \mathbf{\sigma}^* = \begin{bmatrix} \sigma_x^0 & \sigma_y^0 & \tau_{xy}^0 \\ \sigma_x^0 & \sigma_y^0 & \tau_{xy}^0 \\ \tau_{xy}^0 & \tau_{xy}^0 & \tau_{xy}^0 \end{bmatrix} + \begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \end{bmatrix} = \mathbf{S} \mathbf{\beta} + \mathbf{\sigma}^*,$$

(7)

$$\mathbf{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \cdots \\ \beta_N \end{bmatrix}^T,$$

where $\beta_i (i = 1-N)$ are $N$ unknown stress parameters; $\mathbf{S}$ is the stress solution matrix:

$$\mathbf{S} = \begin{bmatrix} \sigma_{x1} & \sigma_{x2} & \sigma_{x3} & \cdots & \sigma_{xk} \\ \sigma_{y1} & \sigma_{y2} & \sigma_{y3} & \cdots & \sigma_{yk} \\ \tau_{xy1} & \tau_{xy2} & \tau_{xy3} & \cdots & \tau_{xyk} \end{bmatrix}_{3 \times k}$$

or $\mathbf{S} = \begin{bmatrix} \sigma_{r1} & \sigma_{r2} & \sigma_{r3} & \cdots & \sigma_{rk} \\ \sigma_{\theta1} & \sigma_{\theta2} & \sigma_{\theta3} & \cdots & \sigma_{\theta k} \\ \tau_{\theta1} & \tau_{\theta2} & \tau_{\theta3} & \cdots & \tau_{\theta k} \end{bmatrix}_{3 \times k}$

(8)

It must be noted that the components (stress interpolation functions) in $\mathbf{S}$, which are not the assumed stresses used in usual hybrid-stress models, are all derived from $k$ fundamental solutions of stress function. $\mathbf{\sigma}^*$ is a particular solution corresponding to body forces, and for constant body forces $b_x$ and $b_y$, it can be given by:
\[ \sigma^* = \begin{bmatrix} \sigma_x^* \\ \sigma_y^* \\ \tau_{xy}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -b_y x - b_x y \end{bmatrix} \]  

(9)

Therefore, these components are also fundamental stress solutions which satisfy all governing equations. This is the key point for developing HSF element models. According to the principle of minimum complementary energy [29–33], we have

\[ \beta = M^{-1} (Hq^* - M^*) \quad \text{with} \quad M^* = \int_{A^e} S^T C^* \sigma^* dA. \]  

(10)

The element nodal equivalent load vectors caused by concentrated and distributed line forces can be determined by the standard procedure for the conventional finite elements. And the element nodal equivalent load vector caused by body forces is given by

\[ p^* = H^T M^{-1} M^* - V^T \quad \text{with} \quad V = \int_{A^e} \sigma^* L^T N dS. \]  

(11)

Once the element nodal displacement vector \( q^e \) is solved, the element stresses can be given by

\[ \sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = SM^{-1} (-M^* + Hq^*) + \sigma^*. \]  

(12)

The stress solutions at any point can be readily obtained by substituting the global coordinates of this point within an element into \( S \) in the above equation, and then, the resulting strains can be obtained by constitutive equations.

### 2.2. 8-Node plane HSF elements [29]

Ref. [29] developed two 8-node plane continuum HSF elements, in which the simpler one, HSF-Q8-15β, will be employed later for simulation of the crack growth. Consider the 8-node quadrilateral elements shown in Fig. 4, any boundary of the element can be either straight or curved, and \((\xi, \eta)\) are the usual isoparametric coordinates. Different with the usual models, the element shapes can be both convex and concave.

The element nodal displacement vector \( q^e \) is given by

\[ q^e = [u_1 \ v_1 \ u_2 \ v_2 \ u_3 \ v_3 \ u_4 \ v_4 \ u_5 \ v_5 \ u_6 \ v_6 \ u_7 \ v_7 \ u_8 \ v_8]^T, \]  

(13)

where \( u_i \) and \( v_i \) \((i = 1–8)\) are the nodal displacements in \( x \)- and \( y \)-directions, respectively.
The displacements along each element boundary are assumed to be quadratic and determined by the displacements on the three nodes of each boundary. Therefore, \( \bar{u}, \bar{v} \) and corresponding matrix \( \mathbf{N} \) (Eq. (6)) of each element boundary can be given as follows:

\[
\begin{align*}
\bar{u}_{\overline{12}} &= \left\{ \begin{array}{c} \bar{u} \\ \bar{v} \end{array} \right\}_{\overline{12}} = \mathbf{N}|_{\eta=-1} \mathbf{q}^e, \\
\bar{u}_{\overline{34}} &= \left\{ \begin{array}{c} \bar{u} \\ \bar{v} \end{array} \right\}_{\overline{34}} = \mathbf{N}|_{\xi=-1} \mathbf{q}^e,
\end{align*}
\]

(14)

where \( \bar{u}_{\overline{12}}, \bar{u}_{\overline{34}}, \bar{u}_{\overline{23}} \) and \( \bar{u}_{\overline{17}} \) are the boundary displacements along element boundaries \( \overline{12} (\eta = -1), \overline{34} (\xi = 1), \overline{23} (\eta = 1) \) and \( \overline{17} (\xi = -1) \), respectively; and

\[
\mathbf{N} = \begin{bmatrix}
N_1^0 & 0 & N_2^0 & 0 & N_3^0 & 0 & N_4^0 & 0 & N_5^0 & 0 & N_6^0 & 0 & N_7^0 & 0 & N_8^0 & 0 \\
0 & N_1^0 & 0 & N_2^0 & 0 & N_3^0 & 0 & N_4^0 & 0 & N_5^0 & 0 & N_6^0 & 0 & N_7^0 & 0 & N_8^0
\end{bmatrix},
\]

(15)

in which \( N_i^0(\xi, \eta) \) (\( i = 1–8 \)) are the shape functions of the standard 8-node isoparametric Serendipity element Q8 and have been given by

\[
N_i^0 = \begin{cases}
-\frac{1}{4} (1 + \xi_i \xi_j)(1 + \eta_i \eta_j)(1 - \xi_i \xi_j - \eta_i \eta_j) & (i = 1, 2, 3, 4), \\
\frac{1}{2} (1 - \xi_i^2)(1 + \eta_i \eta_j) & (i = 5, 7), \\
\frac{1}{2} (1 - \eta_i^2)(1 + \xi_i \xi_j) & (i = 6, 8),
\end{cases}
\]

(16)

where \( (\xi_i, \eta_i) \) are the isoparametric coordinates of node \( i \).

The resulting stress solutions derived from the first fifteen analytical solutions \( \phi_i \) (\( i = 1–15 \)) of the stress function \( \phi \), which are listed in Table 1, are used to construct the matrix \( \mathbf{S} \) in Eqs. (3) and (8). Then, the matrix \( \mathbf{S} \) can be written as

\[
\mathbf{S} = \begin{bmatrix}
\sigma_{x1} & \sigma_{x2} & \sigma_{x3} & \cdots & \sigma_{x15} \\
\sigma_{y1} & \sigma_{y2} & \sigma_{y3} & \cdots & \sigma_{y15} \\
\tau_{xy1} & \tau_{xy2} & \tau_{xy3} & \cdots & \tau_{xy15}
\end{bmatrix}_{3 \times 15},
\]

(17)

It can be seen that the stress fields possess third-order completeness in the global coordinates \( x \) and \( y \).

The detailed integration procedures for evaluating the matrices \( \mathbf{M} \) and \( \mathbf{H} \) (see Eq. (3)) have been given in Appendices of Ref. [29].

### 2.3. A new multi-node plane singular HSF element

Actually, the hybrid-stress function method described in Section 2.1 can be directly used for developing singular HSF element models for analysis of cracks. Here, a new HSF singular element HSF-Crack is constructed.

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Consider a $J$-node (multi-node) arbitrary polygonal singular HSF element shown in Fig. 5, $(x_t, y_t)$ are the Cartesian coordinates of the crack tip and taken as the origin of the polar coordinate system. Node 1 and node $J$ are located at the same position but belong to two different surfaces ($\theta = -\pi$ and $\theta = \pi$, respectively) of the crack.

The element nodal displacement vector $\mathbf{q}^e$ is given by

$$\mathbf{q}^e = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 & u_5 & v_5 & \cdots & u_{J-1} & v_{J-1} & u_J & v_J \end{bmatrix}^T,$$

where $u_i$ and $v_i$ ($i = 1 - J$) are the nodal displacements in $x$- and $y$-directions, respectively.

In order to connect the 8-node HSF element compatibly, the displacements along each element boundary are also assumed to be quadratic and determined by the displacements on the three nodes of each boundary. Therefore, $\bar{u}$, $\bar{v}$ and corresponding matrix $\mathbf{N}$ (Eq. (6)) of the $i$th element boundary can be given as follows:

$$\begin{align*}
\mathbf{u}_{ih} &= \left\{ \bar{u} \right\}_{ih} = \mathbf{N}_{ih} \mathbf{q}^e, \\
\mathbf{N}_{2i-1-2j} &= \begin{bmatrix}
0 & \cdots & 0 & \overline{N}_1 & 0 & \overline{N}_1 & 0 & \cdots & 0
\end{bmatrix}, \\
\mathbf{N}_{2i-2-1j} &= \begin{bmatrix}
0 & \cdots & 0 & \overline{N}_1 & 0 & \overline{N}_1 & 0 & \cdots & 0
\end{bmatrix},
\end{align*}$$

with

$$\begin{align*}
\overline{N}_1 &= -\frac{1}{2}(1 - \eta)\eta, \\
\overline{N}_2 &= 1 - \eta^2, \\
\overline{N}_3 &= \frac{1}{2}(1 + \eta)\eta,
\end{align*}$$

where $\eta$ is the local coordinate along each element boundary, and $-1 \leq \eta \leq 1$.

According to Williams’s analysis of the stress field at a plane crack tip [35], the analytical solutions of stresses, which can satisfy the equilibrium equations and free surface conditions, can be derived from the stress function satisfying following bi-harmonic equation (compatibility equation)

$$\nabla^4 \phi_k = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \phi_k = 0.$$
In the polar coordinate system shown in Fig. 5, the symmetric part of stress functions, which are corresponding to crack problem of Mode I and satisfy Eq. (22), can be written as

\[
\phi_k = r^{2k+1} \left\{ A_k \cos \left( \frac{k}{2} - 1 \right) \theta + B_k \cos \left( \frac{k}{2} + 1 \right) \theta \right\} \quad (k = 1, 2, \ldots). \tag{23}
\]

The resulting stresses are:

\[
\begin{align*}
\sigma_{r\theta} &= \begin{cases}
\frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} & \text{if } k \neq 0, \\
\frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \rho^2} & \text{if } k = 0,
\end{cases} \\
\sigma_{\theta \rho} &= \begin{cases}
\frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial \rho} & \text{if } k \neq 0, \\
\frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \rho^2} & \text{if } k = 0,
\end{cases} \\
\tau_{r\theta} &= \begin{cases}
\frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial \rho} & \text{if } k \neq 0, \\
\frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \rho^2} & \text{if } k = 0,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\sigma_{r\theta} &= r^{2k-1} \left\{ A_k \frac{k}{2} \cos \left( \frac{k}{2} - 1 \right) \theta + B_k \frac{k}{2} + 1 \cos \left( \frac{k}{2} + 1 \right) \theta \right\} \\
\sigma_{\theta \rho} &= r^{2k-1} \left\{ A_k \frac{k}{2} \cos \left( \frac{k}{2} - 1 \right) \theta + B_k \frac{k}{2} + 1 \cos \left( \frac{k}{2} + 1 \right) \theta \right\} \\
\tau_{r\theta} &= r^{2k-1} \left\{ A_k \frac{k}{2} \sin \left( \frac{k}{2} - 1 \right) \theta + B_k \frac{k}{2} + 1 \sin \left( \frac{k}{2} + 1 \right) \theta \right\} \quad (k = 1, 2, \ldots). \tag{24}
\end{align*}
\]

After introducing the following free surface conditions

\[
\sigma_{r\theta}|_{\theta = \pm \pi} = \tau_{r\theta}|_{\theta = \pm \pi} = 0, \tag{25}
\]

we can obtain [24]

\[
B_k = - \frac{A_k}{k^2 + 1} \left[ (-1)^k + \frac{k}{2} \right]. \tag{26}
\]

Let

\[
A_k = \beta_{2k-1} \quad (k = 1, 2, \ldots), \tag{27}
\]

then, the fundamental analytical solutions for stresses of symmetric part (mode I) can be written as

\[
\begin{align*}
\sigma_r &= \beta_{2k-1} \left\{ r^{2k-1} \left( \frac{3}{2} \cos \left( \frac{k}{2} - 1 \right) \theta + \frac{k}{2} + 1 \right) \cos \left( \frac{k}{2} + 1 \right) \theta \right\} \\
\sigma_\theta &= \beta_{2k-1} \left\{ r^{2k-1} \left( \frac{k}{2} + 1 \cos \left( \frac{k}{2} - 1 \right) \theta - \frac{k}{2} + 1 \right) \cos \left( \frac{k}{2} + 1 \right) \theta \right\} \\
\tau_{r\theta} &= \beta_{2k-1} \left\{ r^{2k-1} \left( \frac{k}{2} - 1 \sin \left( \frac{k}{2} - 1 \right) \theta - \frac{k}{2} + 1 \right) \sin \left( \frac{k}{2} + 1 \right) \theta \right\} \quad (k = 1, 2, \ldots). \tag{28}
\end{align*}
\]

It can be seen that the first term \((k = 1)\) possesses the singularity when \(r \to 0\).

Similarly, the antisymmetric part of stress functions, which are corresponding to crack problem of Mode II and satisfy Eq. (22), can be written as

\[
\phi_j = r^{2j+1} \left\{ C_j \sin \left( \frac{j}{2} - 1 \right) \theta + D_j \sin \left( \frac{j}{2} + 1 \right) \theta \right\} \quad (j = 1, 2, \ldots) \tag{29}
\]

As same as the symmetric problem, from the free boundary conditions (25) of crack surface, we can obtain

\[
D_j = - \frac{C_j}{j^2 + 1} \left[ \frac{j}{2} - (-1)^j \right]. \tag{30}
\]

It should be noted that, when \(j = 2\), the stress function \(\phi_2 = 0\), so that the resulting stress solutions will also be zeros. Such solutions must be omitted. Otherwise, the matrix \(M\) in Eq. (3) would be singular and cannot be inversed. Let

\[
C_j = \beta_{2k} \quad (j = 1, 3, 4, 5, \ldots; \ k = 1, 2, 3, 4, \ldots), \tag{31}
\]

then, the fundamental analytical solutions for stresses of antisymmetric part can be written as
is the first stress parameter in Eq. (32). Once the element nodal displacement vector can be determined by Eq. (10).

The stress intensity factor of mode II is established. It can be denoted as

\[ w = s ] \]

where \( M = 0.25 \), respectively. The reference solution for the stress intensity factors are [19,36]

\[ K_I > \frac{2p}{\pi} \]

\[ K_I \approx \left. \frac{2p}{\pi} \right|_{\theta=0} = \sqrt{2\pi} \beta_1, \]

where \( \beta_1 \) is the first stress parameter in Eq. (28). And the stress intensity factor \( K_{II} \) of mode II is

\[ K_{II} = \left. \frac{2p}{\pi} \right|_{\theta=0} = -\sqrt{2\pi} \beta_2, \]

where \( \beta_2 \) is the first stress parameter in Eq. (32). Once the element nodal displacement vector \( \mathbf{q} \) is solved, \( \beta_1 \) and \( \beta_2 \) can be determined by Eq. (10).

In order to evaluate the matrices \( \mathbf{M} \) and \( \mathbf{H} \) (see Eq. (3)), the \( J \)-node polygonal singular element HSF-Crack can be divided into \( \{J-1\}/2 \) sub-triangles \( A1, A2, \ldots, A(J-1)/2 \), as shown in Fig. 6. Their evaluation procedures are given in Appendix A.

3. Performance tests for the singular HSF element

Similar to the sub-region mixed element (SRME) method [24–28] mentioned in Introduction, the new singular element HSF-Crack and the 8-node plane continuum HSF element HSF-Q8-15\( \beta \) [29] can be employed together to calculate crack problems. Here, the singular element HSF-Crack will be collocated on the stress concentration region near the tip of the crack, while the continuum element HSF-Q8-15\( \beta \) are used on the even stress region at the periphery. This may be also an economical scheme for solving crack problems. Before simulation of crack propagation, behaviors of element HSF-Crack with different integration points, shapes, nodes, sizes, etc., must be fully tested and understood.

As shown in Fig. 7, a rectangular cantilever cracked plate under plane strain condition is considered. The plate is subjected to a shear loading \( \tau = 1 \) Mpa uniformly distributed at its top end which leads to a mix-mode crack state. The geometric parameters are: width \( w = 7.0 \) mm, half height \( h = 8.0 \) mm, length of crack \( l_c = 3.5 \) mm, and thickness \( t = 1.0 \) mm. And the Young’s modulus and the Poisson ratio are \( E = 3 \times 10^7 \) MPa and \( \mu = 0.25 \), respectively. The reference solution for the stress intensity factors are [19,36]

\[ K_1 = 3.40 \text{ MPa mm}^{1/2}, \quad K_{II} = 4.55 \text{ MPa mm}^{1/2}. \]

This example is employed to test the performance of the new singular element HSF-Crack.
3.1. Test I: performance under different shapes, Gauss integral schemes, and numbers of trial functions for the singular element

In this test, the performance of the singular element with different shapes, Gaussian integral schemes, and numbers of trial functions will be assessed.

Three different mesh divisions for this test are given by Fig. 8. Only twenty HSF-Q8-15β elements and one 17-node HSF-Crack element, with totally 85 nodes, are used in each mesh. The shape of the singular element HSF-Crack is quite free, and three shape cases are considered: (a) regular octagonal shape; (b) square shape; and (c) equilateral triangular shape. And the Gaussian integration orders for element HSF-Crack are 5, 7 and 16, respectively, i.e., the matrix $M$ will be evaluated by using $5/2C^2$, $7/2C^2$ and $16/2C^2$ Gauss points in each sub-triangle (see Eq. (3) and Appendix A), respectively, while corresponding matrix $H$ be evaluated by using 5, 7 and 16 Gauss points along each boundary (see Eq. (3) and Appendix A). The number of trial functions (see Eqs. (28), (32) and (33)) used by the singular element is changed from 5 to 40.

The normalized values of resulting $K_I$ and $K_{II}$ are listed in Table 2 and plotted in Fig. 9. Excellent results are obtained using quite limit computation cost. It can be concluded that: (i) If node numbers are the same, the shapes of singular element will not influence the results too much, good results can be obtained by all three...
(ii) The number of Gauss points has little influence on the results, convergent and stable results can be obtained when the Gaussian integration order is only 5 (i.e., 5 × 5 Gauss points in each sub-triangle for evaluating \( \mathbf{M} \), and 5 Gauss points along each boundary for evaluating \( \mathbf{H} \)). (iii) When the number of trial functions is more than 20, the results are quite good and stable.

The present scheme is a high-order method for crack problems, so its computation efficiency should be tested. The problem in Test I has also been solved by the X-FEM [5] using 1081 elements with 1152 nodes and 12 to 21 enrich nodes. And its SIF results have been given in Table 2 for comparison. In fact, better SIF results can be obtained by the present method using quite less elements.

Here, we compile the X-FEM [5] into our FEM program, and find its average CPU time for solving this problem is 1.70 s. The program is executed by a personal computer with ‘Inter Core i7-2600K CPU @3.40 GHz’ and 16 GB of memory. The average CPU times spent by the present schemes are plotted in Fig. 10. The relations of the computation efficiency with different Gaussian integral schemes, numbers of trial functions for the singular element, are clearly illustrated. It can be seen that the longest computation time is no more than 0.9 s. Therefore, the HSF elements can provide better results with lower computational costs.

3.2. Test II: performance under different numbers of nodes and trial functions for the singular element

In this test, the performance of the singular element with different numbers of nodes and trial functions will be assessed. 5 × 5 and 5 Gauss points (i.e., the order of Gaussian integration is 5) are employed for evaluating matrices \( \mathbf{M} \) and \( \mathbf{H} \) in each sub-triangles, respectively.

Six different mesh divisions for this test are given in Fig. 11, in which square and triangular singular element HSF-Crack with different number of nodes are employed. Meshes I, II and III employ square HSF-Crack elements with 13, 17 and 21 nodes, respectively, while Meshes IV, V and VI employ triangular HSF-Crack elements with 13, 17 and 21 nodes, respectively. The normalized values of resulting \( K_I \) and \( K_{II} \) are listed in Table 3 and plotted in Fig. 12.
From Table 3 and Fig. 12, one can find that: (i) For meshes with 17- or 21-node square and triangular HSF-Crack elements, the results are quite good (the relative errors are less than 1%) and stable when the number of trial functions is more than 20; (ii) For meshes with 13-node square and triangular HSF-Crack elements, the maximum relative errors reach 2% or so when the number of trial functions is more than 20.

### 3.3. Test III: performance under different sizes and numbers of trial functions for the singular element

In this test, the performance of the singular element with different sizes and numbers of trial functions will be assessed. The order of Gaussian integration for evaluating matrices \( M \) and \( H \) in each sub triangle is still 5. Only triangular HSF-Crack elements with different sizes are considered.

As shown in Fig. 13, \( a \) is the length of crack that covered by HSF-Crack element. In all meshes given by Fig. 13 (also in Figs. 8 and 11), the crack tip is assumed to be located at the center of the HSF-Crack element. So, \( 2a \) can be treated as a characteristic length which denotes the size of the HSF-Crack element. However, it should be noted that, the crack tip can be located in any position within the singular element.

Meshes I, II and III, given by Fig. 13(a)–(c), respectively, use the same numbers of elements and nodes. And the characteristic length \( 2a \) varies from 4 mm to 0.25 mm. From Table 4 and Fig. 14, it can be seen that good and stable results can be obtained by using Meshes I and II when the number of trial functions is more than 20, while Mesh III cannot provide acceptable results. In fact, the reason why Mesh III performs badly is not due to the small size of the singular element. It can be found that, in Meshes I and II, the largest boundary length ratios of elements around the singular element are 4.206 and 3.125, respectively; but this value for Mesh

| Table 2 | Normalized results of \( K_I \) and \( K_{II} \) for Test I (Fig. 8). |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| No. of trial functions (selected) | 5 | 10 | 20 | 25 | 30 | 40 |
| (a) Normalized values of \( K_I \), the reference solution is \( K_I = 34.0 \text{ MPa mm}^{1/2} \) |
| Mesh I octagonal | 5 × 5 Gauss points | 0.9376 | 0.9945 | 0.9968 | 0.9960 | 0.9957 | 0.9960 |
| 7 × 7 Gauss points | 0.9377 | 0.9947 | 0.9969 | 0.9961 | 0.9957 | 0.9961 |
| 16 × 16 Gauss points | 0.9377 | 0.9947 | 0.9970 | 0.9961 | 0.9958 | 0.9961 |
| Mesh II square | 5 × 5 Gauss points | 0.9285 | 0.9900 | 0.9954 | 0.9950 | 0.9956 | 0.9957 |
| 7 × 7 Gauss points | 0.9286 | 0.9901 | 0.9955 | 0.9951 | 0.9957 | 0.9958 |
| 16 × 16 Gauss points | 0.9286 | 0.9901 | 0.9956 | 0.9951 | 0.9957 | 0.9959 |
| Mesh III triangular | 5 × 5 Gauss points | 0.9663 | 1.0058 | 1.0021 | 0.9969 | 0.9995 | 0.9968 |
| 7 × 7 Gauss points | 0.9664 | 1.0058 | 1.0022 | 0.9969 | 0.9996 | 0.9970 |
| 16 × 16 Gauss points | 0.9664 | 1.0059 | 1.0023 | 0.9970 | 0.9997 | 0.9971 |
| Singular ES-FEM [19] | | | | | | |
| X-FEM [5] | | | | | | |
| Wavelet Galerkin X-FEM [37] | | | | | | |
| (b) Normalized values of \( K_{II} \), the reference solution is \( K_{II} = 4.55 \text{ MPa mm}^{1/2} \) |
| Mesh I octagonal | 5 × 5 Gauss points | 0.9934 | 0.9963 | 0.9968 | 0.9969 | 0.9970 | 0.9970 |
| 7 × 7 Gauss points | 0.9934 | 0.9963 | 0.9969 | 0.9969 | 0.9970 | 0.9971 |
| 16 × 16 Gauss points | 0.9934 | 0.9963 | 0.9969 | 0.9969 | 0.9970 | 0.9971 |
| Mesh II square | 5 × 5 Gauss points | 0.9811 | 0.9938 | 0.9967 | 0.9973 | 0.9974 | 0.9974 |
| 7 × 7 Gauss points | 0.9811 | 0.9938 | 0.9967 | 0.9973 | 0.9975 | 0.9974 |
| 16 × 16 Gauss points | 0.9811 | 0.9938 | 0.9967 | 0.9973 | 0.9975 | 0.9974 |
| Mesh III triangular | 5 × 5 Gauss points | 0.9937 | 0.9931 | 0.9974 | 0.9988 | 0.9993 | 1.0001 |
| 7 × 7 Gauss points | 0.9937 | 0.9931 | 0.9974 | 0.9988 | 0.9993 | 0.9996 |
| 16 × 16 Gauss points | 0.9937 | 0.9931 | 0.9975 | 0.9988 | 0.9993 | 0.9996 |
| Singular ES-FEM [19] | | | | | | |
| X-FEM [5] | | | | | | |
| Wavelet Galerkin X-FEM [37] | | | | | | |
III is 27.314. This kind of mesh distortion may influence the results greatly for a small singular element whose boundaries are quite close to the crack tip. If the ratio is reduced by adding layers of elements, such as the Mesh IV (Fig. 13(d)) in which the largest boundary length ratio of the elements around the singular element is only 3.278, excellent results can be again obtained (see Fig. 14 and Table 4).

Fig. 9. Results for Test I: performance under different shapes, Gauss integral schemes, and number of trial functions for the singular element.
3.4. Test IV: convergence test for a centre curved crack problem

As shown in Fig. 15, a square plate with a centre curved crack is considered. This plate is under plane strain conditions, and subjected to a uniformly distributed tension \( \sigma = 1.0 \). The edge length \( L = 160.0 \), and the

Fig. 10. Computation times for Test I: time efficiency under different shapes, Gauss integral schemes, and number of trial functions for the singular element.

3.4. Test IV: convergence test for a centre curved crack problem

As shown in Fig. 15, a square plate with a centre curved crack is considered. This plate is under plane strain conditions, and subjected to a uniformly distributed tension \( \sigma = 1.0 \). The edge length \( L = 160.0 \), and the
thickness $t = 1.0$. The Young’s modulus and the Poisson ratio are $E = 3 \times 10^7$ and $\mu = 0.25$, respectively. And the stress intensity factors (SIF) for the infinite body are given by [5]
Table 3
Normalized results of $K_I$ and $K_{II}$ for Test II (Fig. 10).

<table>
<thead>
<tr>
<th>No. of trial functions (selected)</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>(a) Normalized values of $K_I$, the reference solution is $K_I = 34.0 \text{ MPa mm}^{1/2}$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mesh I  Square 13-node</td>
<td>0.9195</td>
<td>0.9875</td>
<td>0.9845</td>
<td>0.9813</td>
<td>0.9796</td>
<td>0.9795</td>
</tr>
<tr>
<td>Mesh II Square 17-node</td>
<td>0.9286</td>
<td>0.9901</td>
<td>0.9955</td>
<td>0.9951</td>
<td>0.9957</td>
<td>0.9958</td>
</tr>
<tr>
<td>Mesh III Square 21-node</td>
<td>0.9296</td>
<td>0.9900</td>
<td>0.9963</td>
<td>0.9966</td>
<td>0.9965</td>
<td>0.9967</td>
</tr>
<tr>
<td>Mesh IV Triangular 13-node</td>
<td>0.9592</td>
<td>0.9866</td>
<td>0.9810</td>
<td>0.9799</td>
<td>0.9818</td>
<td>0.9793</td>
</tr>
<tr>
<td>Mesh V  Triangular 17-node</td>
<td>0.9664</td>
<td>1.0058</td>
<td>1.0022</td>
<td>0.9969</td>
<td>0.9996</td>
<td>0.9970</td>
</tr>
<tr>
<td>Mesh VI Triangular 21-node</td>
<td>0.9633</td>
<td>0.9962</td>
<td>0.9952</td>
<td>0.9907</td>
<td>0.9921</td>
<td>0.9949</td>
</tr>
<tr>
<td><strong>(b) Normalized values of $K_{II}$, the reference solution is $K_{II} = 4.55 \text{ MPa mm}^{1/2}$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mesh I  Square 13-node</td>
<td>0.9697</td>
<td>0.9900</td>
<td>0.9927</td>
<td>0.9914</td>
<td>0.9923</td>
<td>0.9925</td>
</tr>
<tr>
<td>Mesh II Square 17-node</td>
<td>0.9811</td>
<td>0.9938</td>
<td>0.9967</td>
<td>0.9973</td>
<td>0.9975</td>
<td>0.9974</td>
</tr>
<tr>
<td>Mesh III Square 21-node</td>
<td>0.9799</td>
<td>0.9931</td>
<td>0.9962</td>
<td>0.9968</td>
<td>0.9970</td>
<td>0.9969</td>
</tr>
<tr>
<td>Mesh IV Triangular 13-node</td>
<td>0.9940</td>
<td>0.9844</td>
<td>0.9877</td>
<td>0.9861</td>
<td>0.9833</td>
<td>0.9823</td>
</tr>
<tr>
<td>Mesh V  Triangular 17-node</td>
<td>0.9937</td>
<td>0.9931</td>
<td>0.9974</td>
<td>0.9988</td>
<td>0.9993</td>
<td>0.9996</td>
</tr>
<tr>
<td>Mesh VI Triangular 21-node</td>
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<td>0.9960</td>
<td>0.9995</td>
<td>0.9978</td>
<td>0.9982</td>
<td>0.9976</td>
</tr>
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</table>

Fig. 12. Results for Test II: performance under different numbers of nodes and trial functions for the singular element.
20 HSF-Q8-15β elements, and one triangular HSF-Crack element with \( a = 0.125 \) and 17 nodes.

20 HSF-Q8-15β elements, and one triangular HSF-Crack element with \( a = 1 \) and 17 nodes.

20 HSF-Q8-15β elements, and one triangular HSF-Crack element with \( a = 0.125 \) and 17 nodes.

36 HSF-Q8-15β elements, and one triangular HSF-Crack element with \( a = 0.125 \) and 17 nodes.

Fig. 13. Meshes for Test III: performance under different sizes and number of trial functions for the singular element.

Table 4
Normalized results of \( K_I \) and \( K_{II} \) for Test III (Fig. 12).

<table>
<thead>
<tr>
<th>No. of trial functions (selected)</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Normalized values of ( K_I ), the reference solution is ( K_I = 34.0 \text{ MPa mm}^{1/2} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mesh I</td>
<td>0.8885</td>
<td>1.0075</td>
<td>1.0003</td>
<td>0.9951</td>
<td>0.9957</td>
<td>0.9952</td>
</tr>
<tr>
<td>Mesh II</td>
<td>0.9664</td>
<td>1.0058</td>
<td>1.0022</td>
<td>0.9969</td>
<td>0.9996</td>
<td>0.9970</td>
</tr>
<tr>
<td>Mesh III</td>
<td>0.5792</td>
<td>0.5751</td>
<td>0.8237</td>
<td>0.8047</td>
<td>0.9306</td>
<td>1.0143</td>
</tr>
<tr>
<td>Mesh IV</td>
<td>0.9996</td>
<td>1.0009</td>
<td>1.0000</td>
<td>0.9949</td>
<td>0.9988</td>
<td>0.9972</td>
</tr>
<tr>
<td>(b) Normalized values of ( K_{II} ), the reference solution is ( K_{II} = 4.55 \text{ MPa mm}^{1/2} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mesh I</td>
<td>0.9760</td>
<td>0.9653</td>
<td>0.9968</td>
<td>0.9974</td>
<td>0.9981</td>
<td>0.9974</td>
</tr>
<tr>
<td>Mesh II</td>
<td>0.9937</td>
<td>0.9931</td>
<td>0.9974</td>
<td>0.9988</td>
<td>0.9993</td>
<td>0.9996</td>
</tr>
<tr>
<td>Mesh III</td>
<td>0.7311</td>
<td>0.8636</td>
<td>1.0292</td>
<td>1.0467</td>
<td>1.0738</td>
<td>1.0862</td>
</tr>
<tr>
<td>Mesh IV</td>
<td>0.9937</td>
<td>0.9972</td>
<td>0.9991</td>
<td>0.9993</td>
<td>0.9997</td>
<td>1.0012</td>
</tr>
</tbody>
</table>

Fig. 14. Results for Test III (Fig. 13): performance under different sizes and number of trial functions for the singular element.
\[ K_I = \frac{\sigma}{2} (\pi R \sin(\beta))^{1/2} \left[ \frac{1 - \sin^2(\beta/2) \cos^2(\beta/2)}{1 + \sin^2(\beta/2)} + \cos(3\beta/2) \right] \]

\[ K_{II} = \frac{\sigma}{2} (\pi R \sin(\beta))^{1/2} \left[ \frac{1 - \sin^2(\beta/2) \cos^2(\beta/2)}{1 + \sin^2(\beta/2)} + \sin(3\beta/2) \right] \]

where \( R \) is the radius of the circular arc and \( 2\beta \) is the subtended angle of the arc.
Due to symmetry, only a half plate is modeled. The crack is a circular arc centred at (0.0, 3.75) with radius \( R = 4.25 \). The arc extends from (0.0, 0.5) on the symmetry boundary to (2.0, 0.0). The subtended angle \( \beta = 28.0725^\circ \). Thus, the stress intensity factors for the infinite body with this crack geometry are \( K_I = 2.014568 \) and \( K_{II} = 1.111589 \).

Three meshes with 69 (247 nodes), 113 (391 nodes) and 161 (547 nodes) elements are given in Fig. 16, in which the latter two meshes are the refined schemes of the first one. Most part of the two surfaces of the curved crack is directly discretized by the 8-node continuum HSF-Q8-15\( ^{\beta} \) elements with curved sides, while the crack tip is modeled by one triangular 17-node singular HSF-Crack element. Since the path of the crack covered in the HSF-Crack element is a straight line, the HSF-Crack element should not be too large. Here, the characteristic lengths of the triangular HSF-Crack element used in the three meshes are \( 2a = 0.2, 2a = 0.1 \) and \( 2a = 0.05 \), respectively. Furthermore, the orders of Gaussian integration for the elements HSF-Crack and HSF-Q8-15\( ^{\beta} \) are both 5, and 27 trial functions are adopted for HSF-Crack.

The normalized \( K_I \) and \( K_{II} \) values and the average CPU times used by present method are given in Table 5. It can be seen that, with the refinement of the mesh, the present scheme exhibit good convergence and stability. The results obtained by X-FEM [5] and corresponding computation time (provided by the authors’ program and computer same as those in Section 3.1) are also given for comparison.

### Table 5

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Present</th>
<th>Refine</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of nodes</td>
<td>Mesh I</td>
<td>Mesh II</td>
<td>Mesh III</td>
</tr>
<tr>
<td>CPU time (second)</td>
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<td>0.685</td>
<td>1.065</td>
</tr>
<tr>
<td>Normalized ( K_I )</td>
<td>1.0242</td>
<td>1.0188</td>
<td>1.0100</td>
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<tr>
<td>X-FEM [5]</td>
<td>No. of nodes</td>
<td>1037 (with 54 enrich nodes)</td>
<td>1.725</td>
</tr>
<tr>
<td>CPU time (second)</td>
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<td>0.685</td>
<td>1.065</td>
</tr>
<tr>
<td>Normalized ( K_I )</td>
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<td>1.0107</td>
<td>1.0100</td>
</tr>
<tr>
<td>Present</td>
<td>Mesh I</td>
<td>Mesh II</td>
<td>Mesh III</td>
</tr>
<tr>
<td>No. of nodes</td>
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<td>391</td>
<td>547</td>
</tr>
<tr>
<td>CPU time (second)</td>
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<tr>
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<td>1.0010</td>
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<td>X-FEM [5]</td>
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<td>1.725</td>
</tr>
<tr>
<td>CPU time (second)</td>
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<td>0.685</td>
<td>1.065</td>
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<tr>
<td>Normalized ( K_{II} )</td>
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</tbody>
</table>

### 3.5. Summary of using the singular HSF-Crack element

The performance of the singular HSF-Crack has been carefully assessed by above tests and other tests which are not given in this paper. It can be seen that excellent results can be obtained by HSF-Crack element with quite limited computation costs. In order to use this model correctly, following points should be emphasized.
For most cases, once the number of trial functions is more than 20, good and stable results can be obtained. Therefore, a number between 20 and 30 can be selected as the number of trial functions used for constructing the finite element formulations.

Fig. 17. Crack propagation paths for the single-edge cracked plate (Section 4.1).
Fig. 18. PMMA specimen with three holes (Section 4.2).

(a) First step:
129 elements,
with 448 nodes

(b) 16th step:
120 elements,
with 425 nodes

(c) 27th step:
110 elements,
with 395 nodes

(d) 41th (last) step:
135 elements,
with 480 nodes

Fig. 19. Crack propagation path for the PMMA specimen (Case I).
(b) $5 \times 5$ and 5 Gauss points are quite enough for evaluating matrices $M$ and $H$ (see Eq. (3) and Appendix A) in each sub triangles, respectively.

(c) The shape of HSF-Crack element is quite free. Triangular, rectangular and other polygonal shapes can all produce good results.

(d) The number of element nodes should not be less than 13. A 17-node HSF-Crack element with triangular, or square, or octagonal shapes is an appropriate model for most cases.

(e) The largest edge length ratios of the elements around the singular HSF-Crack element should not be more than 5, especially for the small HSF-Crack element whose boundaries are quite close to the crack tip.

4. Quasi-static simulations on crack Propagation

The tests in last section have showed the performance of the new singular element HSF-Crack, together with the plane 8-node element HSF-Q8-15β, in computations of crack problem. Excellent results can be obtained under relatively lower computational costs. In this section, these models are used to simulate...
quasi-static crack propagation. Several classical examples, which are all under plane strain state, are solved by the present models. According to the results and conclusions given by the last section, the simulation strategies used in this section are as follows.

(i) **Elements.** At the crack-tip, two singular element cases are considered: 13- and 17-node triangular HSF-Crack elements. Both elements adopt 27 trial functions, $5 \times 5$ Gauss points for evaluating $M$, and $5$ Gauss points for $H$. And plane 8-node HSF-Q8-15 elements are located at the other domain. Furthermore, the size of the elements near the crack-tip should satisfy Point (e) given by Section 3.4.

(ii) **Meshing and remeshing.** One structure can be divided two kinds of regions. (1) Region I: fixed mesh domain. The regions far away from the cracks are modeled using coarse and fixed meshes (using HSF-Q8-15 elements). The meshes in these domains will not change during the simulation procedure unless the crack penetrates into them. Once any crack enters into Region I, Region I will change to Region II. It should be noted that Region I is not necessary for the present method. (2) Region II: remeshing domain, the regions in which the cracks grow. After each propagate step of the crack, the meshes in these domain (between the boundaries of the Region I and the boundaries of the singular HSF-Crack element) will be remeshed (using HSF-Q8-15 elements) automatically. It should be noted that, the meshing scheme is quite simple: the simple free meshing and seeding techniques for quadrilateral elements, which are employed by SIMULIA-Abaqus [38], are quite enough. Although distorted elements may appear, it will not influence the results since element HSF-Q8-15 is immune to severe mesh distortions.

(iii) **Crack propagation direction.** According to the discussions given by Ref. [6], the direction angle of the crack propagation for each increment step is determined by

$$
\theta_c = 2 \arctan \left[ \frac{1}{4} \left( \frac{K_I}{K_H} - \text{sign}(K_H) \times \sqrt{8 + \left( \frac{K_I}{K_H} \right)^2} \right) \right].
$$

(38)

4.1. Single-edge cracked plate under mixed-mode loading

The initial state of this example is the same as the test used in the last section and shown in Fig. 7. All geometric and material parameters have been given in Section 3. The length of the crack increment at each propagation step is fixed to $d_{lc} = 0.1$ mm. Two singular element cases are considered: (i) 13-node triangular
Fig. 22. Dual crack propagation problem in a square plate with two holes (Section 4.3).

Fig. 23. Dual crack propagation paths, $d_{lc} = 0.1$ mm (Section 4.3).
HSF-Crack element; (ii) 17-node triangular HSF-Crack element. The characteristic length of the singular element is \( 2a = 2d_{lc} = 0.2 \text{ mm} \), and this length remains unchanged during the whole simulation procedure. Once one part of the singular element exceeds the boundary of the structure, the structural boundary and the other part of the singular element will form a new singular element with new shape and nodes.

After 36 propagation steps, the crack will penetrate the whole structure. It is found that, the total number of elements used in each step never exceeds 60. Fig. 17 (a) and (b) show the meshes and the crack growth path at first, 18th and 36th (the last) steps. Fig. 17(c) plots the final crack paths obtained by the two singular element cases. And the CPU time, including both computation and remeshing time, for each step is also given in Fig. 17. It can be seen that the present schemes can produce excellent results under relatively low computational expense.

4.2. Crack propagation in a PMMA specimen with three holes

Experimental results for crack propagation in a PMMA specimen have been obtained by Bittencourt et al. [39], in which the presence of holes in the plates disturbs the stress/strain fields providing interesting curvilinear crack trajectories. The geometry parameters given in Fig. 18 are \( l_c = 1 \text{ mm} \), \( b = 4 \text{ mm} \) for Case I, and \( l_c = 1.5 \text{ mm} \), \( b = 5 \text{ mm} \) for Case II. And the material constants of the specimen are: \( E = 3 \times 10^8 \text{ MPa} \) and \( \mu = 0.3 \). In the simulations, the length of the crack propagation step \( dl_c \) is fixed to 0.05 mm on condition that \( |K_{II}/K_I| \geq 0.2 \), otherwise \( dl_c \) is fixed to 0.1 mm [19]. Only 17-node triangular singular HSF-Crack element case is considered in this example. The character length of the singular element \( 2a = 2d_{lc} \) for most steps. But it will decrease to a smaller value when the crack is very close to the structural boundary.
The crack propagates 41 steps in Case I and 38 steps in Case II. Figs. 19–21 show the crack propagation paths obtained by present scheme. The present results agree with the reference solutions given by Bittencourt [39]. And the results obtained by other methods [21,39,40–42] are also plotted in Fig. 21 for comparison.

4.3. Dual crack propagation in a square plate with two holes

Fig. 22 shows a square plate with cracks emanating from two holes, subjected to a far-field tension. The Young’s modulus and the Poisson’s ratio of the plate are $E = 10^5$ MPa, $\mu = 0.3$. In the initial configuration, both cracks have a length of 0.1 mm and are oriented at angles $\theta = 45^\circ$ and $-45^\circ$ for the left and right holes, respectively. In the simulation, two incremental length cases of the crack are considered: (1) $d_l = 0.1$ mm; and (2) $d_l = 0.05$ mm. The 17-node triangular singular HSF-Crack element is again employed for simulating two crack tips. The largest character length of the singular element is $2d_l$, and this length will reduce to a smaller value when one of the crack tips is very close to the other crack or the structural boundaries.

The total numbers of crack propagation steps for above two cases are 8 and 16, respectively. Figs. 23 and 24 show the crack growth paths at different steps obtained by present scheme. It can be seen that, the two cracks grow in a nearly symmetrical pattern for both cases, although the mesh configurations are not symmetric. Fig. 25(a) shows that, the final crack trajectories obtained by present scheme using two different lengths of propagation step agree with each other. And Fig. 25(b) shows the present results agree well with those obtained by X-FEM [6].
5. Concluding remarks

Following the concept of the hybrid stress-function element method proposed by Refs. [29–33], a new shape-free multi-node singular hybrid stress-function (HSF) element, HSF-Crack, is developed in this paper. The analytical solutions of stress function for crack problem are used as the trial functions, and the element stiffness matrix is established by the principle of minimum complementary energy.

Then, this new model HSF-Crack and the 8-node shape-free plane HSF element HSF-Q8-15β [29] are employed to simulate the quasi-static 2D crack propagation problems. Compared with other well-known methods, such new scheme exhibits four advantages: (i) the shape and the number of element nodes can be flexibly adjusted as required; (ii) high precision for stress intensity factors (SIF) can be obtained due to the advantages of the HSF method; (iii) only simple remeshing with a very coarse mesh is enough in each simulation step; (iv) unstructured mesh containing extremely distorted elements can be used without losing precision. Numerical examples show that excellent results can be obtained under relatively low computational costs.

This paper demonstrates that the proposed scheme, in which the singular element HSF-Crack and the plane element HSF-Q8-15β are used together, is an effective way for dealing with quasi-static crack propagation problem. It may be generalized to dynamic fracture analysis, and this will be discussed in near future. Furthermore, on high-order approximation with exact geometry, it is also very promising to further investigate this method in combination with isogeometric analysis [43–48].

Acknowledgments

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Appendix A

As shown in Fig. 8, in order to evaluate matrices $\mathbf{M}$ and $\mathbf{H}$ (see Eq. (2)) of the present element HSF-Crack by standard Gaussian numerical integral procedure, the $J$-node polygonal singular element HSF-Crack is divided into $(J-1)/2$ sub-triangles $A_1, A_2, \ldots, A_{(J-1)/2}$. And the polar coordinates $(r, \theta)$ should be transformed into isoparametric coordinates $(\xi, \eta)$.

Firstly, within any sub-triangle of element HSF-Crack, $r$ can be expressed in terms of the global Cartesian coordinates $(x, y)$ by

$$r = \sqrt{(x-x_t)^2 + (y-y_t)^2},$$

(A1)

where $(x_t, y_t)$ are the Cartesian coordinates of the crack tip.

Secondly, according to the rational intervals of $\theta$ ($\in[-\pi, \pi]$) and the angle $\psi$ ($\in[-\pi, \pi]$) between the crack direction and the $x$-axis (see Fig. 8), $\theta$ can be expressed in terms of the global Cartesian coordinates $(x, y)$ by following procedure:

$$\theta = \begin{cases} \hat{\theta} - 2\pi & \pi < \hat{\theta} \leq 2\pi \\ \hat{\theta} & -\pi \leq \hat{\theta} \leq \pi \\ \hat{\theta} + 2\pi & -2\pi \leq \hat{\theta} < -\pi \end{cases}$$

with $\tilde{\theta} = \hat{\theta} - \psi$, \quad $\hat{\theta} = \begin{cases} \tilde{\theta} & x \geq x_t, \\ \tilde{\theta} - \text{sign}(\pi, \tilde{\theta}) & x < x_t, \end{cases}$

(A2a)

$$\text{sign}(\pi, \tilde{\theta}) = \begin{cases} \pi & \tilde{\theta} \geq 0 \\ -\pi & \tilde{\theta} < 0 \end{cases}, \quad \tilde{\theta} = \arctan\left(\frac{y-y_t}{x-x_t}\right).$$

(A2b)
Thirdly, any sub-triangle can be treated as a degenerated quadrangle with five nodes, in which the first node and the last node are in coincidence with each other at the crack tip. So, within one sub-triangle, the Cartesian coordinates \((x, y)\) can be interpolated by the isoparametric coordinates \((\xi, \eta)\) as follows

\[
\begin{align*}
x &= \sum_{j=1}^{5} N_j^0(\xi, \eta)x_j, \\
y &= \sum_{j=1}^{5} N_j^0(\xi, \eta)y_j,
\end{align*}
\]  

(A3)

where

\[
\begin{align*}
(x_1, y_1) &= (x_3, y_3) = (x_r, y_r), \\
(x_2, y_2) &= (x_{2-1}, y_{2-1}), \\
(x_3, y_3) &= (x_2, y_2), \\
(x_4, y_4) &= (x_{2+1}, y_{2+1}),
\end{align*}
\]  

(A4)

and

\[
\begin{align*}
N_1^0 &= \frac{1}{2}(1 - \xi)(1 - \eta), \\
N_2^0 &= -\frac{1}{4}\eta(1 + \xi)(1 - \eta), \\
N_3^0 &= \frac{1}{2}(1 + \xi)(1 + \eta)(1 - \eta), \\
N_4^0 &= \frac{1}{4}\eta(1 + \xi)(1 + \eta), \\
N_5^0 &= \frac{1}{4}(1 - \xi)(1 + \eta).
\end{align*}
\]  

(A5)

Then, after substituting Eqs. (A1)–(A5) into the matrix \(S\) given in Eq. (30), \(S\) becomes

\[
S(r, \theta) = S(x, y) = S(\xi, \eta).
\]  

(A6)

Thus, matrix \(M\) for element HSF-Crack can be rewritten as

\[
M = \sum_{i=1}^{(J-1)/2} M_i = \sum_{i=1}^{(J-1)/2} \int_{A_i} S(r, \theta)^T CS(r, \theta) dA = \sum_{i=1}^{(J-1)/2} \left( \int_{-1}^{1} \int_{-1}^{1} S(\xi, \eta)^T CS(\xi, \eta) \left| \frac{J}{d\xi d\eta} \right| \right) A_i,
\]  

(A7)

where \(|J|\) is the is the Jacobian determinant. Then, a Gaussian integration scheme can be used for evaluation of Eq. (A7). The suggested numbers of Gauss points for HSF-Crack are 5 \(\times\) 5.

And the evaluation of matrix \(H\) (Eq. (2)) for element HSF-Crack should be performed along element boundaries \((\xi = 1)\):

\[
H = \sum_{i=1}^{(J-1)/2} H_i = \sum_{i=1}^{(J-1)/2} \int_{\Gamma_i} S(r, \theta)^T L^T N \left| \frac{J}{d\xi d\eta} \right| ds.
\]  

(A8)

Since the components of matrix \(S\) for element HSF-Crack are expressed in terms of polar coordinates, and

\[
\begin{align*}
\sigma_x &= \begin{bmatrix} I_{11}I_{11} & I_{12}I_{12} & -2I_{11}I_{12} \\ I_{01}I_{01} & I_{02}I_{02} & -2I_{01}I_{02} \end{bmatrix}, \\
\sigma_y &= \begin{bmatrix} I_{11}I_{11} & I_{12}I_{12} & -2I_{11}I_{12} \\ I_{01}I_{01} & I_{02}I_{02} & -2I_{01}I_{02} \end{bmatrix}, \\
\tau_{xy} &= \begin{bmatrix} I_{11}I_{11} & I_{12}I_{12} & -2I_{11}I_{12} \\ I_{01}I_{01} & I_{02}I_{02} & -2I_{01}I_{02} \end{bmatrix},
\end{align*}
\]  

(A9)

with

\[
I_{11} = -l_{02} = \cos(\theta + \psi), \quad I_{12} = l_{01} = \sin(\theta + \psi),
\]  

(A10)

the matrix \(L\) in Eq. (A8) is different with \(L\) given by Eq. (4), and must be modified as

\[
L = L_1L_2 = \begin{bmatrix} 1 & 0 & m \\ 0 & 1 & l \\ m & l & 1 \end{bmatrix} \begin{bmatrix} I_{11}I_{11} & I_{12}I_{12} & -2I_{11}I_{12} \\ I_{01}I_{01} & I_{02}I_{02} & -2I_{01}I_{02} \\ l_{11}I_{11} & l_{12}I_{12} & -l_{11}l_{12} - l_{01}l_{02} \end{bmatrix},
\]  

(A11)

where \(l\) and \(m\) are the direction cosines of the outer normal \(n\) of the element boundaries, and given by

\[
l = \frac{dy}{ds}, \quad m = -\frac{dx}{ds}.
\]  

(A12)
Along each boundary ($\xi = 1$), the relation between $d_s$ and $d_\eta$ are given by

$$
\begin{align*}
  d_s &= \left[ \left( \frac{dx}{d\eta} \right)^2 + \left( \frac{dy}{d\eta} \right)^2 \right]^{1/2}, \\
  \xi &= 1
\end{align*}
$$

(A13)

Thus, substitution of Eqs. (7), (17), (30), (A10)–(A13) into Eq. (A8) yields

$$
\begin{align*}
  H &= \sum_{i=1}^{J-1/2} H_i = \sum_{i=1}^{J-1/2} \int_{\Gamma_i} S(1, \eta) (\bar{L}_1 L_2)^T N_{\text{nh}} \, d\eta,
\end{align*}
$$

(A14)

where $N_{\text{nh}}$ is given by Eq. (17); and

$$
\begin{align*}
  \bar{L}_1|_{\xi=1} &= \begin{bmatrix}
  \frac{dx}{d\eta} & 0 & -\frac{dx}{d\eta} \\
  0 & -\frac{dy}{d\eta} & \frac{dy}{d\eta}
\end{bmatrix},
\end{align*}
$$

(A15)

Five Gauss points are suggested for evaluating Eq. (A14).

References