A reciprocity relationship for random diffuse fields in acoustics and structures with application to the analysis of uncertain systems

Robin Langley

UNIVERSITY OF CAMBRIDGE

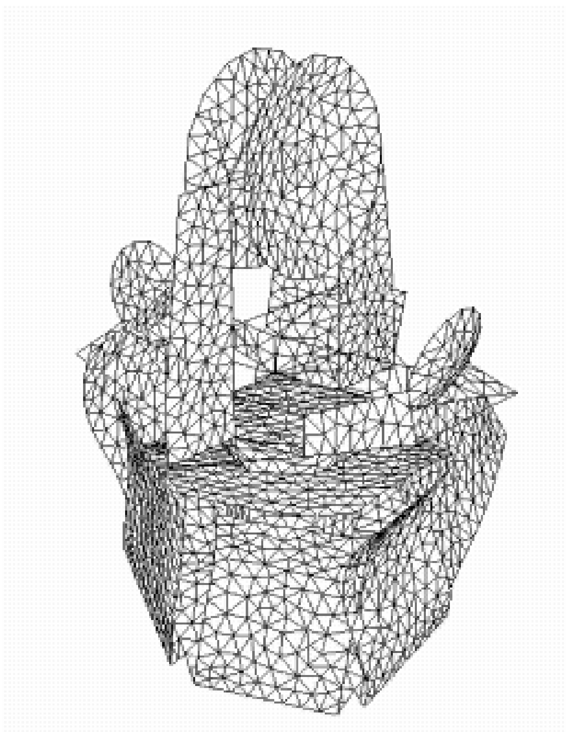
DYNAMICS and VIBRATION

Example 1: Acoustic loads on satellites

Arianne 5 Launch

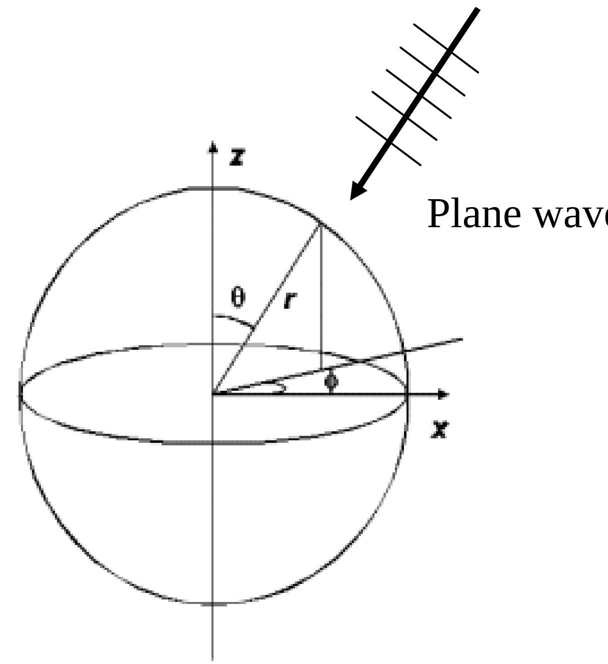
Acoustic test facility at ESTEC

Satellite computational model



Structural model

Pressure p



Diffuse acoustic field

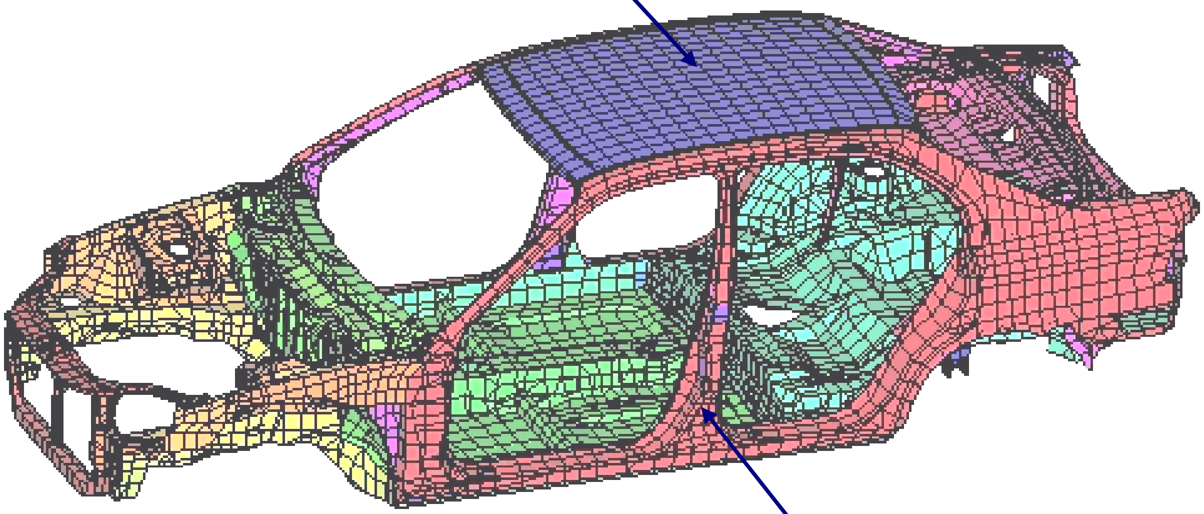
$$p = \sum_j \sum_k \alpha_{jk} \exp\left\{ikr\sin\theta_j\cos\phi_k + ikr\sin\theta_j\sin\phi_k + ikr\cos\theta_j + i\omega t\right\}$$

Pressure modelled as a summation of plane waves at each frequency

This is very time consuming - is there a more efficient approach?

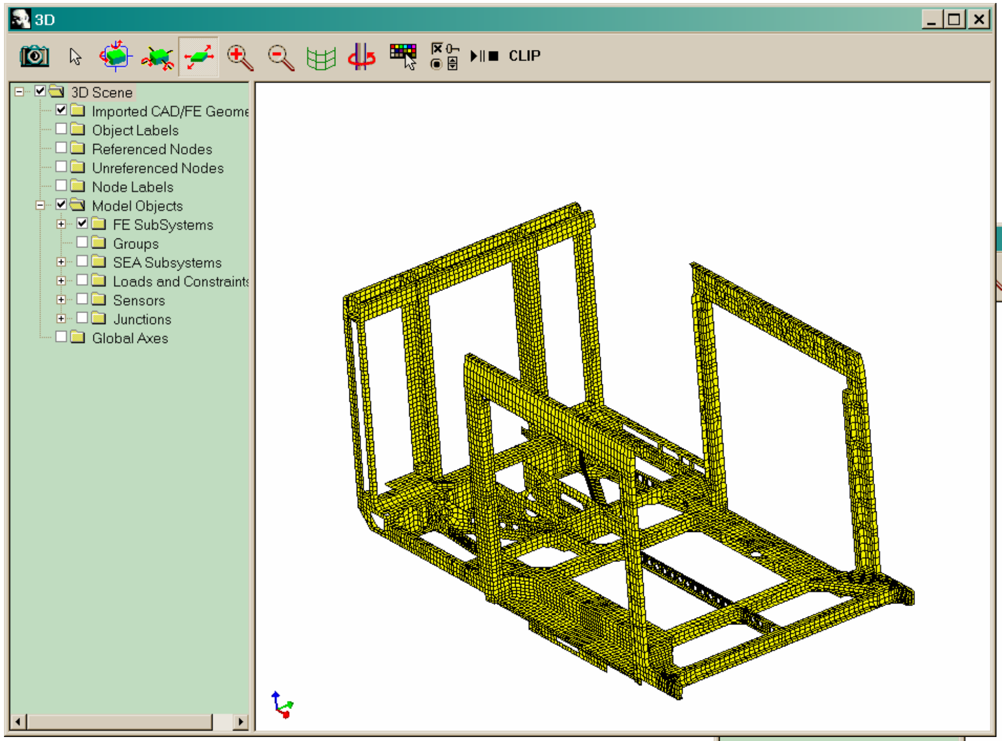
Example 2: High frequency structural vibrations

Thin panels – very many modes, response can be viewed as a diffuse wave field

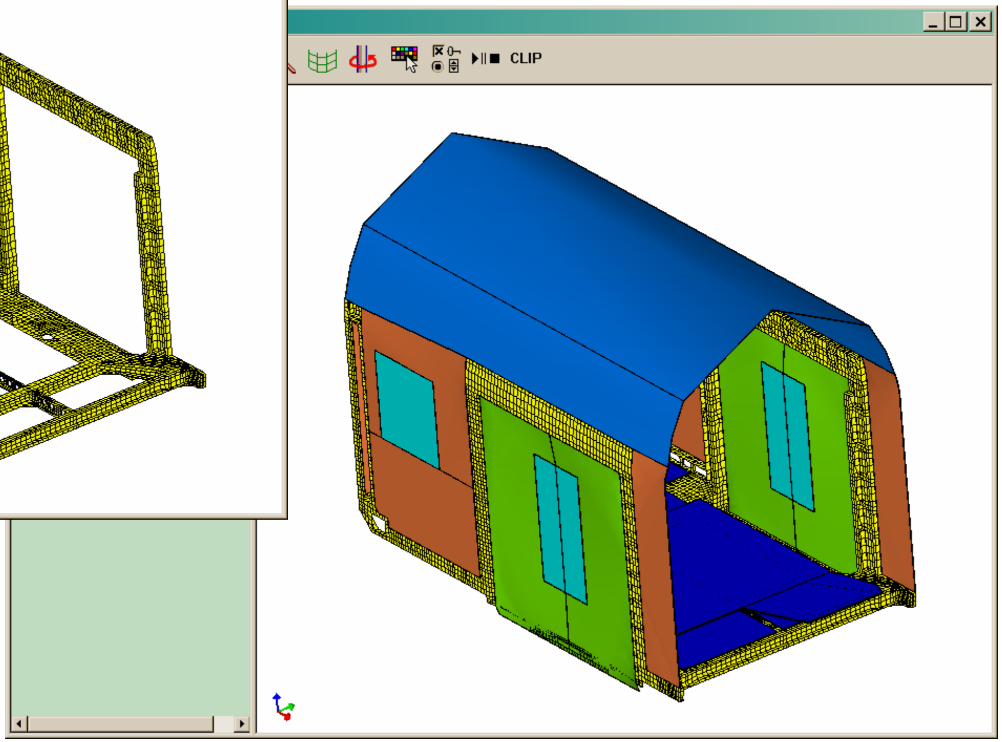


Stiff components – detailed FE model needed

Structure computational model



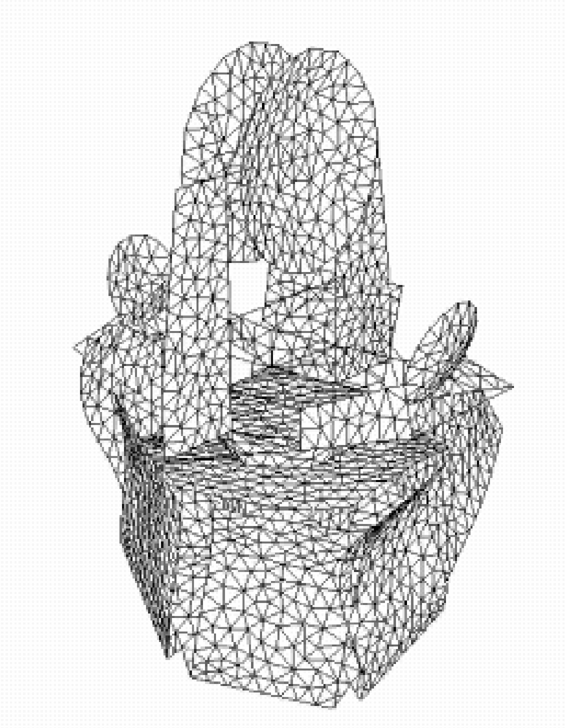
FE model – displacement dofs



SEA model – diffuse wave fields

How do we couple these very different types of model?
Again we need to compute the forces arising from the diffuse waves.

The diffuse field reciprocity relation



Finite element model has dof $\mathbf{x}$
Pressure p produces force vector $\mathbf{f}$
Can we avoid summing waves to compute $\mathbf{f}$?

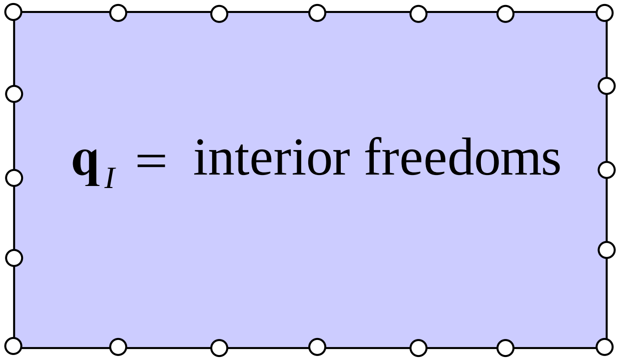
The *diffuse field reciprocity relation* states:

$$\mathbf{S}_{ff} = \mathrm{E}\left[\mathbf{f}\mathbf{f}^{*\mathrm{T}}\right] = \left(\frac{4E}{\omega\pi n}\right)\mathrm{Im}\left\{\mathbf{D}_{\infty}\right\}$$

E is the energy of the diffuse field
n is the modal density of the diffuse field
(for acoustics E/n is specified by the noise level)

$\mathbf{D}_{\infty}$ is the dynamic stiffness matrix for radiation
into an *infinite* acoustic system – readily computable

Short proof of the relation (structural component)



$$\begin{pmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{12}^{\mathrm{T}} & \mathbf{D}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{q}_B \\ \mathbf{q}_I \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \mathbf{0} \end{pmatrix}$$

Express $\mathbf{q}_I$ in terms of blocked modes

$$\mathbf{q}_I = \mathbf{\Phi a}$$

where $\mathbf{a}$ are the modal amplitudes

$$\begin{pmatrix} \mathbf{D}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12}^{\mathrm{T}} & \mathbf{\Lambda} \end{pmatrix} \begin{pmatrix} \mathbf{q}_B \\ \mathbf{a} \end{pmatrix} = \begin{pmatrix} \mathbf{g} \\ \mathbf{0} \end{pmatrix}$$

$$\Lambda_{kk} = \omega_k^2 (1 + i\eta) - \omega^2$$

Boundary dynamic stiffness $\{\mathbf{D}_{11} - \mathbf{C}_{12}\mathbf{\Lambda}^{-1}\mathbf{C}_{12}^{\mathrm{T}}\}\mathbf{q}_B = \mathbf{g}$ or $\mathbf{D}\mathbf{q}_B = \mathbf{g}$

Force on boundary due to interior waves $\mathbf{f} = -\mathbf{C}_{12}\mathbf{a} \quad \Rightarrow \quad \mathbf{f}\mathbf{f}^{*\mathrm{T}} = \mathbf{C}_{12}\mathbf{a}\mathbf{a}^{*\mathrm{T}}\mathbf{C}_{12}^{\mathrm{T}}$

Ensemble averaging

Assume that:

-the natural frequencies are random
-The modal amplitudes **a** correspond to a diffuse field, i.e. they are statistically independent and each mode stores equal energy

$$\mathbf{D} = \mathbf{D}_{11} - \mathbf{C}_{12}\boldsymbol{\Lambda}^{-1}\mathbf{C}_{12}^{\mathrm{T}} \quad \Rightarrow \quad \mathrm{E}[\mathrm{Im}(\mathbf{D})] = \mathrm{E}[\mathbf{C}_{12}\mathbf{C}_{12}^{*\mathrm{T}}](\pi n / 2N\omega)$$

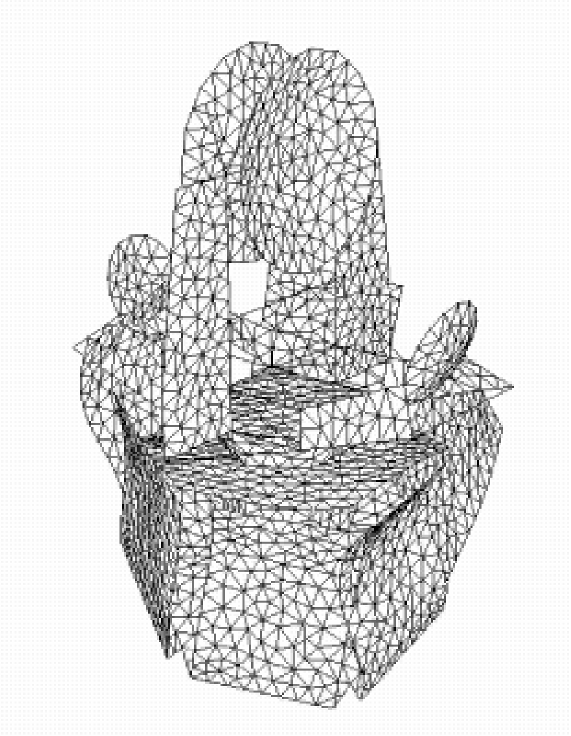
$$\mathbf{f}\mathbf{f}^{*\mathrm{T}} = \mathbf{C}_{12}\mathbf{a}\mathbf{a}^{*\mathrm{T}}\mathbf{C}_{12}^{\mathrm{T}} \quad \Rightarrow \quad \mathrm{E}[\mathbf{f}\mathbf{f}^{*\mathrm{T}}] = \mathrm{E}[\mathbf{C}_{12}\mathbf{C}_{12}^{\mathrm{T}}](2E / N\omega^2)$$

Thus: $$\mathrm{E}\left[\mathbf{f}\mathbf{f}^{*\mathrm{T}}\right] = \left(\frac{4E}{\omega\pi n}\right)\mathrm{E}[\,\mathrm{Im}\{\mathbf{D}\}]$$

The proof is completed by noting that E[**D**]=$\mathbf{D}_\infty$

Smith, P. JASA 34, 640-647, 1962
Shorter, P.J. and Langley, R.S. JASA 117, 85-95, 2005

Application 1: diffuse acoustic loading



$\mathbf{D}_{\infty}$ is the *radiation* dynamic stiffness of the structure. This can be found by using the *boundary element method.*

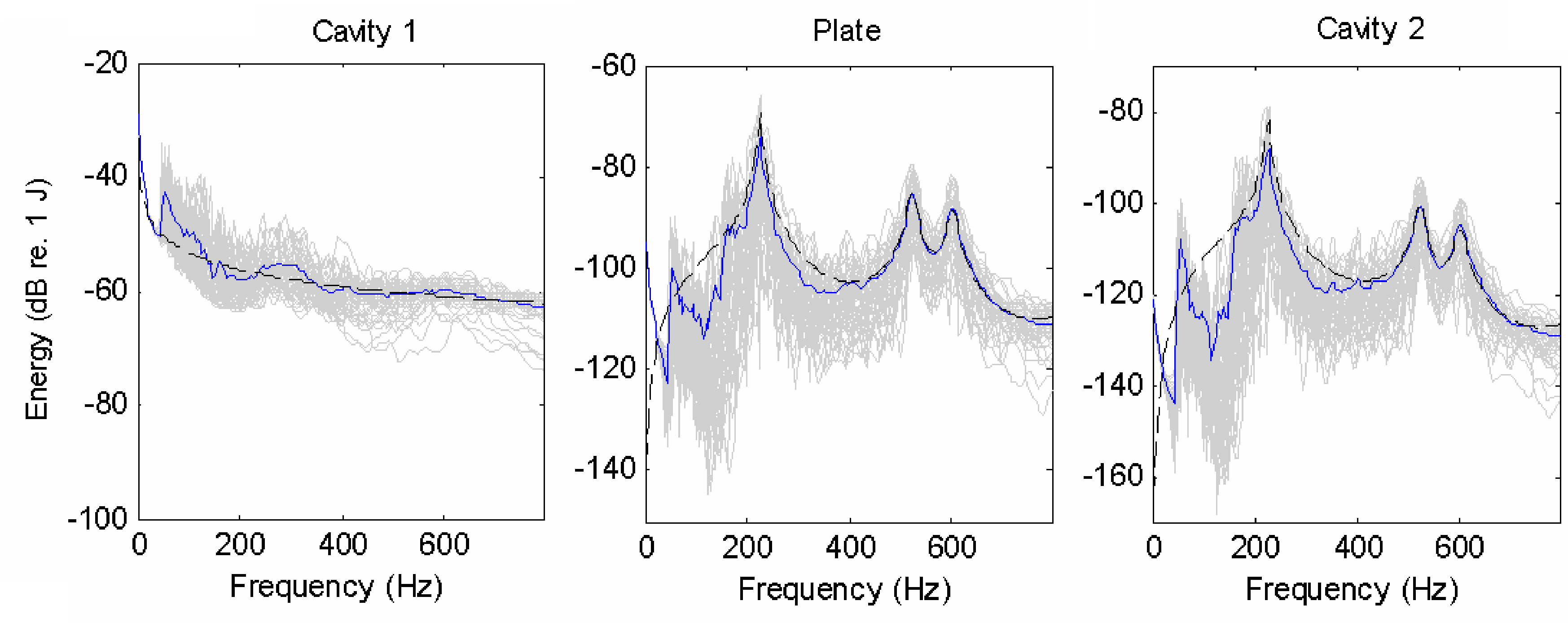
The diffuse field reciprocity relation

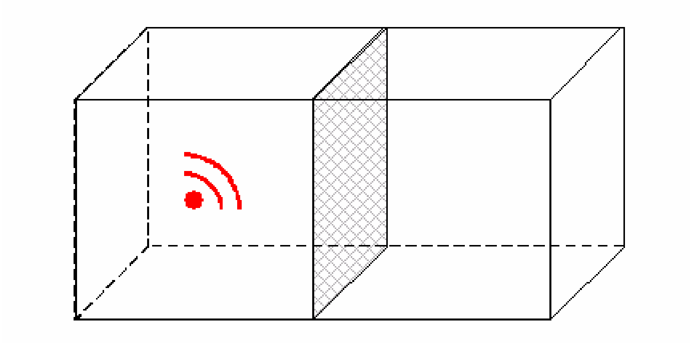
$$\mathrm{E}\left[\mathbf{f}\mathbf{f}^{*\mathrm{T}}\right] = \left(\frac{4E}{\omega\pi n}\right)\mathrm{E}[\,\mathrm{Im}\{\mathbf{D}\}]$$

replaces the need for a direct summation

$$p = \sum_j \sum_k \alpha_{jk} \exp\left\{ikr\sin\theta_j\cos\phi_k + ikr\sin\theta_j\sin\phi_k + ikr\cos\theta_j + i\omega t\right\}$$

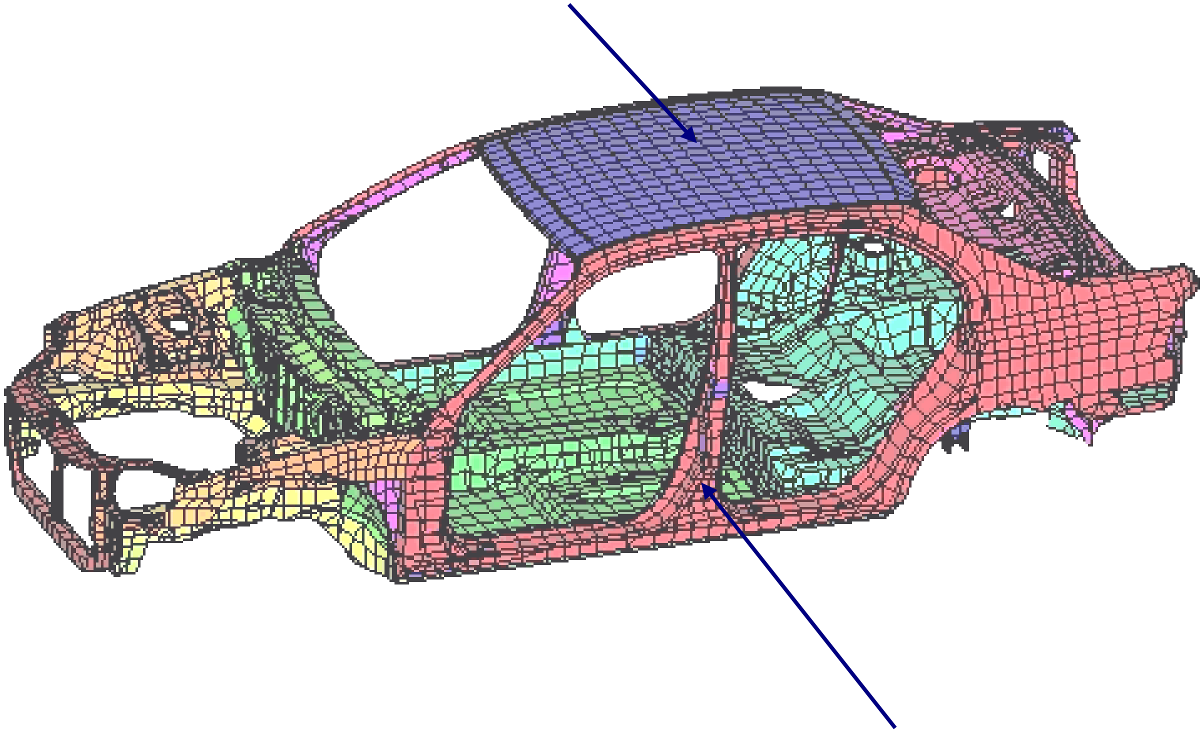
Example – acoustic transmission, ensemble of random cavities





Application 2: the hybrid FE-SEA method

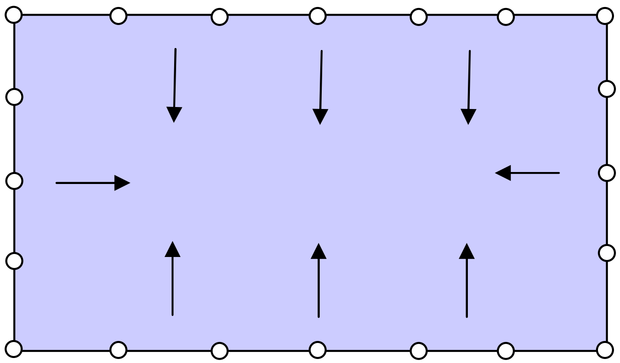
Thin panels – very many modes, response can be viewed as a diffuse wave field. SEA degree of freedom E_k



Stiff components – detailed FE model needed
FE degrees of freedom $\mathbf{q}$

Hybrid Equations – the response of subsystem k

Direct Field



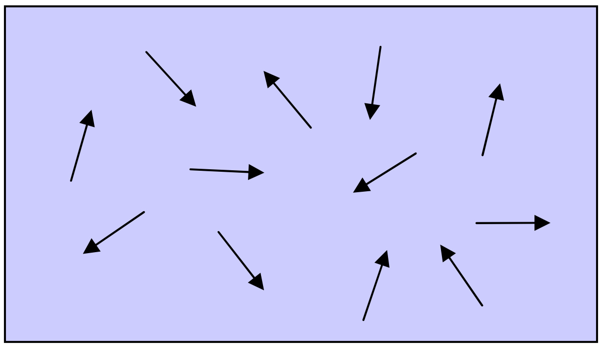
Boundary freedoms q

Boundary motions generate a "direct field" of ingoing waves. Associated boundary forces written as:

$$-\mathbf{D}_{dir}^{(k)}\mathbf{q}$$

\+

Reverberant Field



Reflections produce the "reverberant field". Associated boundary forces written as:

$$\mathbf{f}_{rev}^{(k)}$$

Hybrid Equations continued

So the system equations are written as:

$$\mathbf{D}_{tot}\mathbf{q} = \mathbf{f} + \sum_k \mathbf{f}_{rev}^{(k)}, \qquad \mathbf{D}_{tot} = \mathbf{D}_d + \sum_k \mathbf{D}_{dir}^{(k)}$$

$\mathbf{D}_{dir}^{(k)}$ can be found by considering wave radiation into a semi-infinite system

$\mathbf{f}_{rev}^{(k)}$ is accounted for by using a *diffuse field reciprocity* result:

$$\mathbf{S}_{ff}^{(k),rev} \equiv \mathrm{E}\left[\mathbf{f}_{rev}^{(k)}\mathbf{f}_{rev}^{(k)*T}\right] = \left(\frac{4E_k}{\omega\pi n_k}\right)\mathrm{Im}\left\{\mathbf{D}_{dir}^{(k)}\right\}$$

Final form of the Hybrid Equations

$$\omega(\eta_j + \eta_{d,j})E_j + \sum_k \omega\eta_{jk} n_j (E_j / n_j - E_k / n_k) = P_{in,j}^{ext}$$

$$\mathbf{S}_{qq} = \mathbf{D}_{tot}^{-1}\left[\mathbf{S}_{ff} + \sum_k \left(\frac{4E_k}{\omega\pi n_k}\right)\mathrm{Im}\left\{\mathbf{D}_{dir}^{(k)}\right\}\right]\mathbf{D}_{tot}^{-1*\mathrm{T}}$$

where:

$$P_{in,j}^{ext} = (\omega/2)\sum_{rs}\mathrm{Im}\left\{D_{dir,rs}^{(j)}\right\}\left(\mathbf{D}_{tot}^{-1}\mathbf{S}_{ff}\mathbf{D}_{tot}^{-1*\mathrm{T}}\right)_{rs}$$

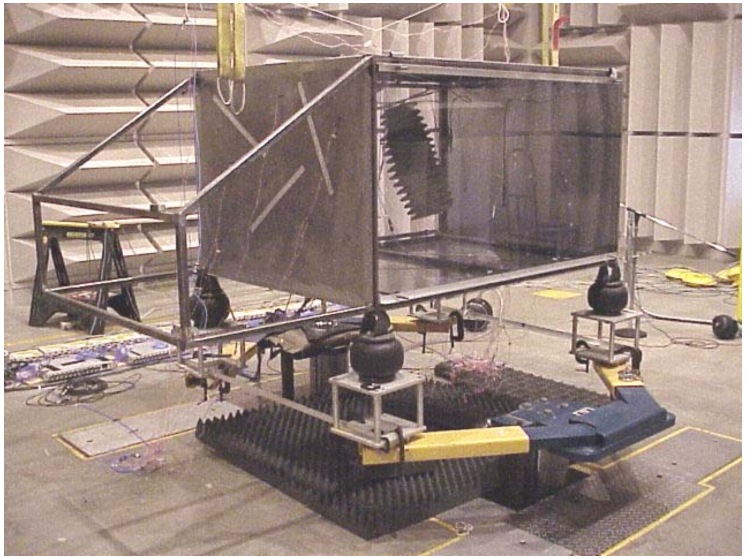
$$\omega\eta_{jk} n_j = (2/\pi)\sum_{rs}\mathrm{Im}\left\{D_{dir,rs}^{(j)}\right\}\left(\mathbf{D}_{tot}^{-1}\,\mathrm{Im}\left\{\mathbf{D}_{dir}^{(k)}\right\}\mathbf{D}_{tot}^{-1*\mathrm{T}}\right)_{rs}$$

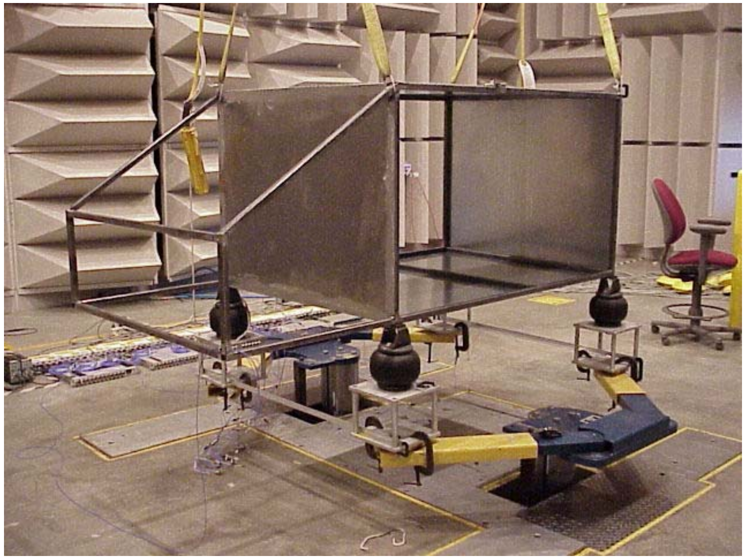
$$\omega\eta_{d,j} = \left(\frac{2}{\pi n_j}\right)\sum_{rs}\mathrm{Im}\left\{D_{d,rs}\right\}\left(\mathbf{D}_{tot}^{-1}\,\mathrm{Im}\left\{\mathbf{D}_{dir}^{(j)}\right\}\mathbf{D}_{tot}^{-1*\mathrm{T}}\right)_{rs}$$

Example application of the hybrid method

Test box

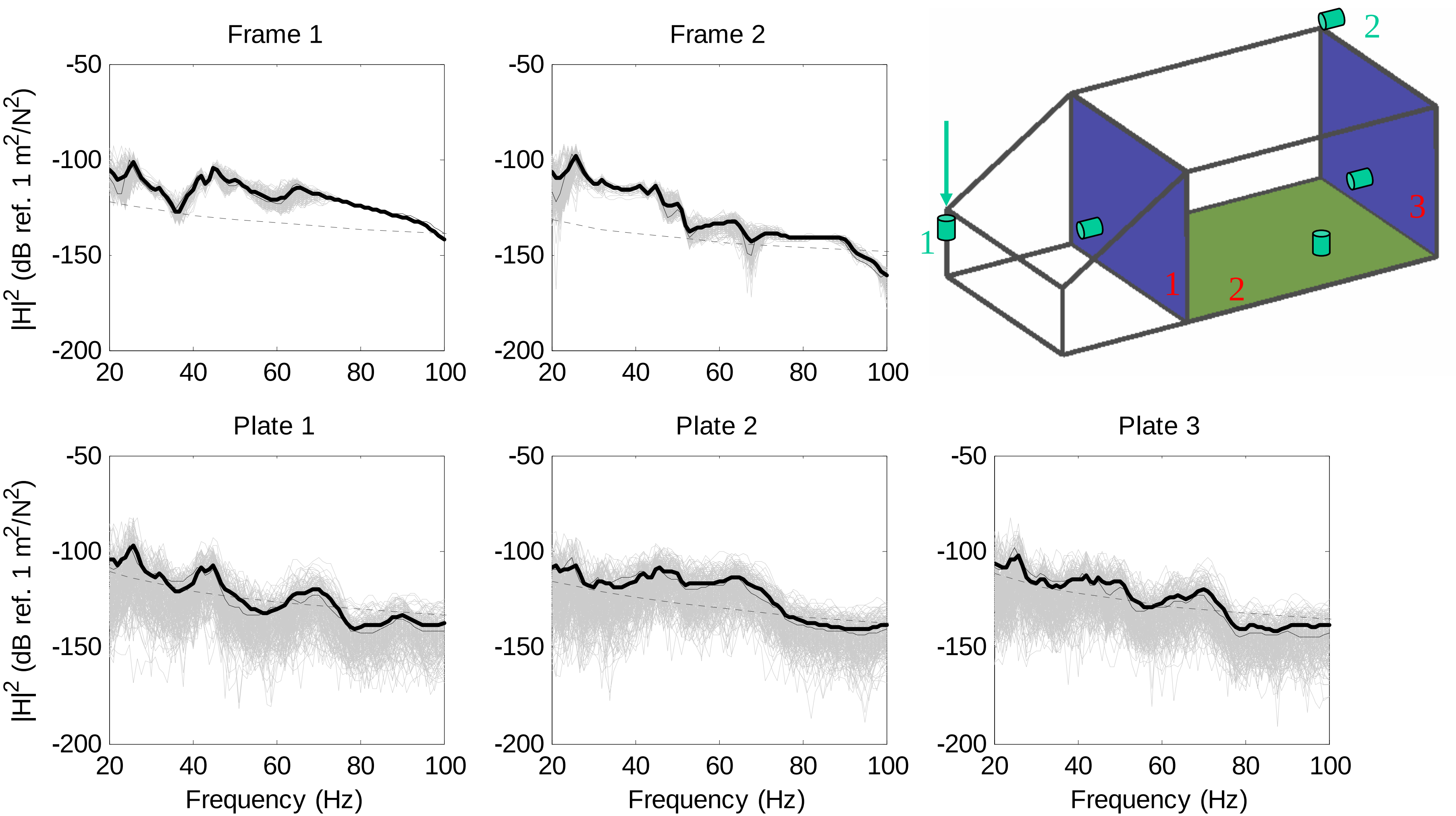
- frame: welded beam sections 1”x1”x1/8”
- 5 plates of different thickness





Courtesy of General Motors Corporation

Example application of the hybrid FE-SEA method



Further Development: Variance Prediction in the Hybrid Method

- Research example: two plates coupled through 3 points

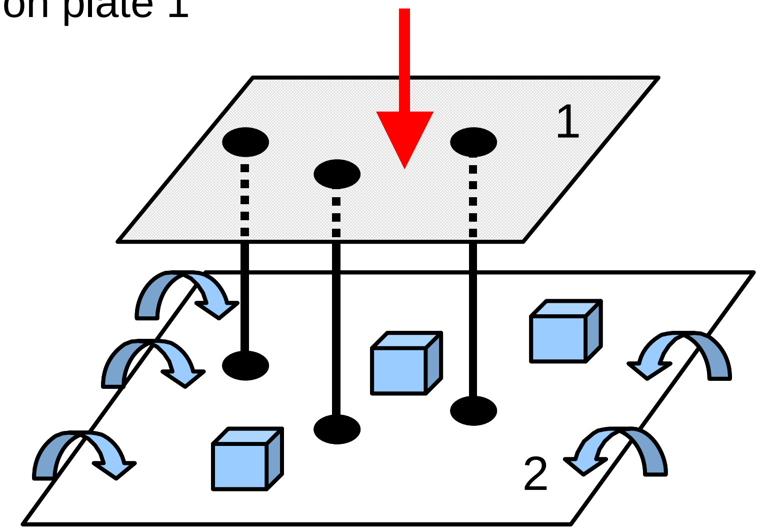
- Plate 1 is deterministic

  (long wavelength, ~ 9 modes below 100 Hz)

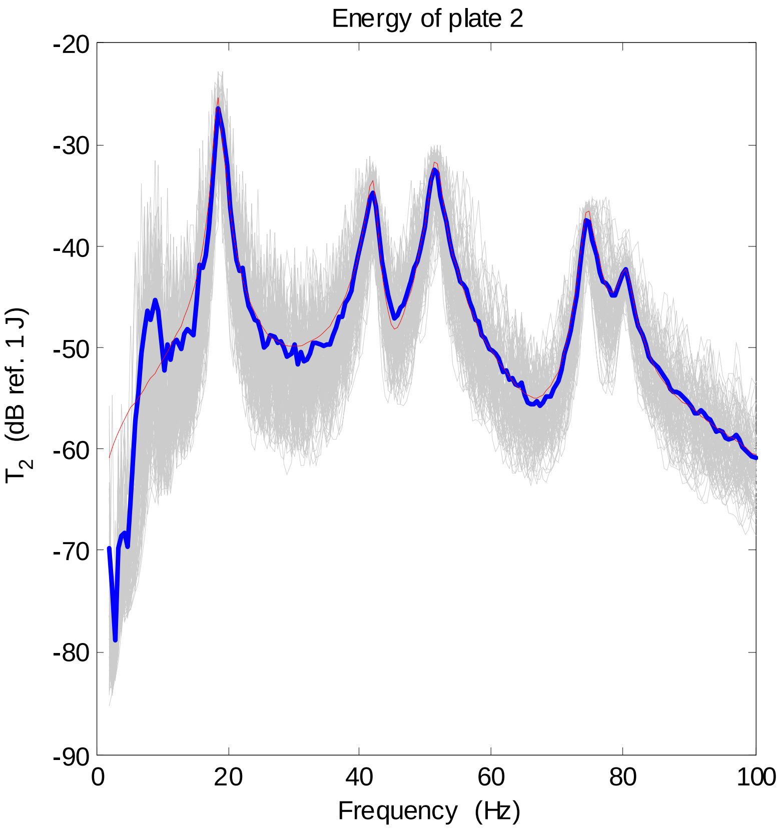
- Plate 2 is randomized by masses and edge springs

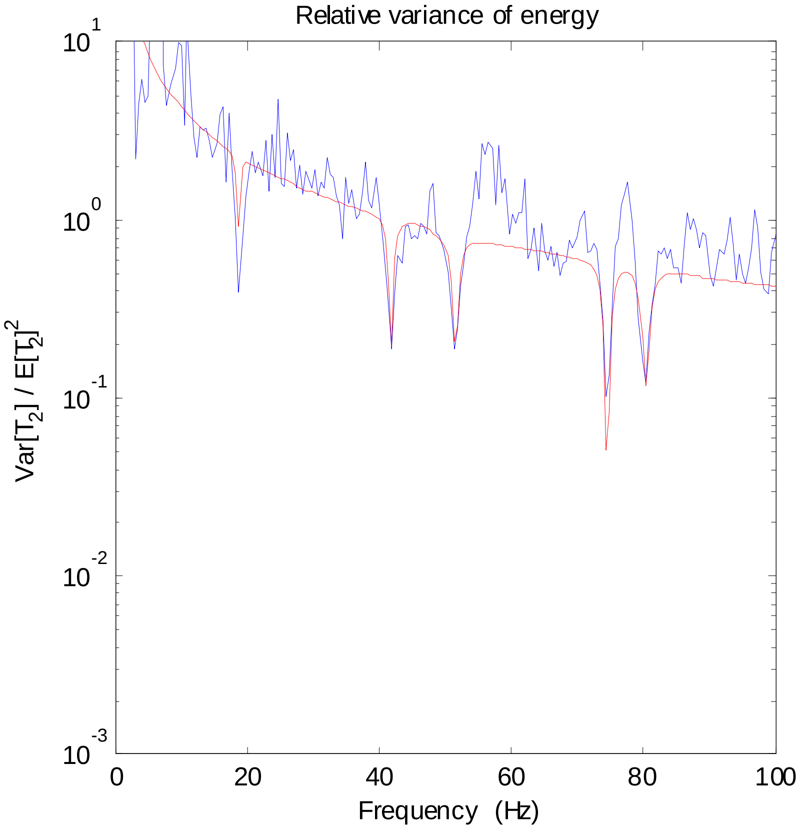
  (short wavelength, ~ 148 modes below 100 Hz)

- Harmonic point force on plate 1

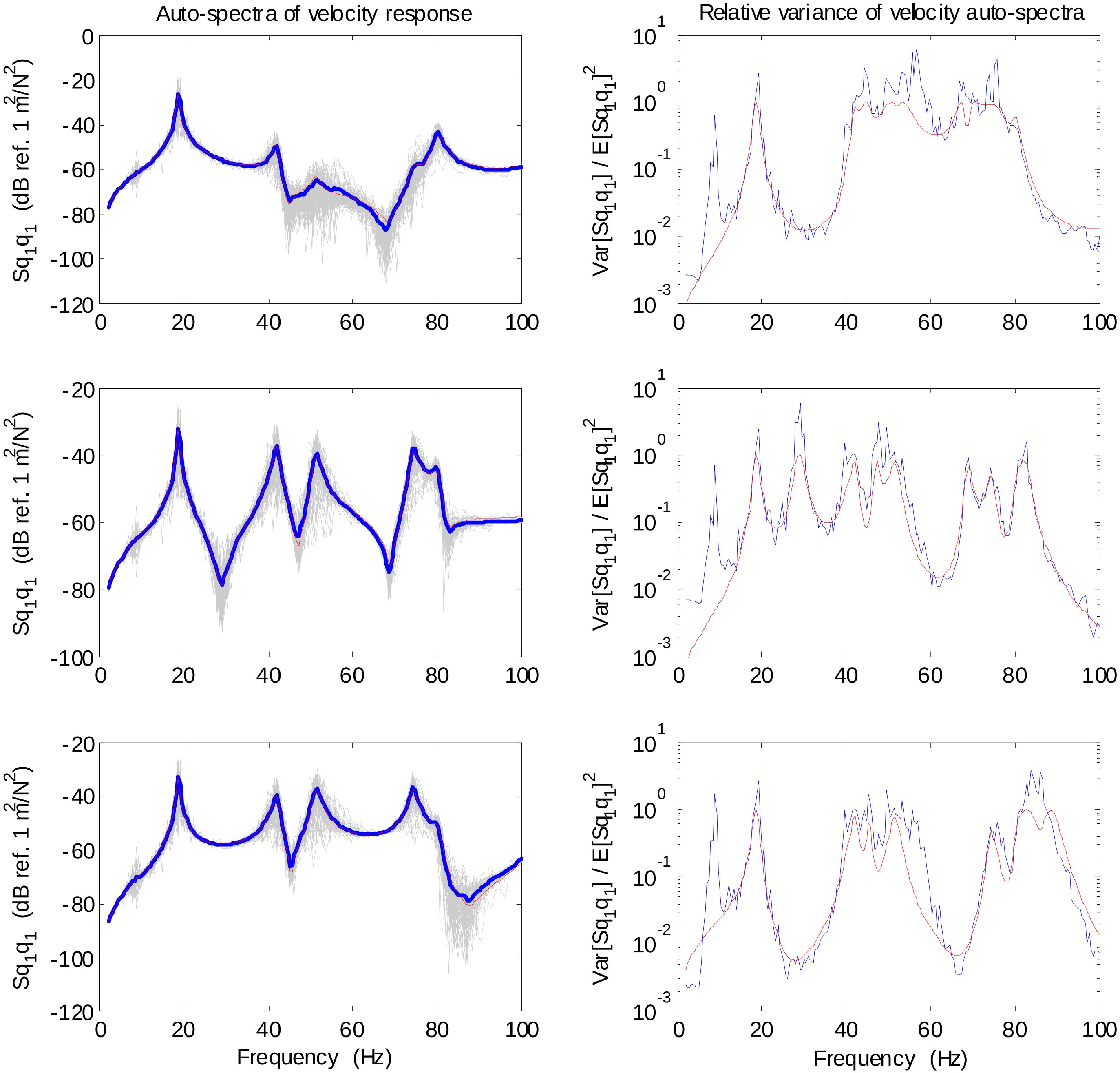


Ensemble mean & relative variance of plate 2 energy response





Ensemble mean & relative variance auto-spectra response at 3 points on plate 1



Postcript

For systems that are driven purely at the boundary, it is unlikely that the excitation will produce a perfect diffuse field. In this case it has recently been found that the reciprocity relationship must be extended:

$$\mathrm{E}[\mathbf{ff}^{*\mathrm{T}}] = \left(\frac{4E}{\pi\omega n}\right)\mathrm{Im}(\mathbf{D}_{\infty}) + \left(\frac{2}{\pi m}\right)\left\{2\,\mathrm{Re}(\mathbf{S}_{\hat{f}\hat{f}}) + q(m)\mathbf{S}_{\hat{f}\hat{f}}\right\}$$

This extension clears up an apparent anomaly between the reciprocity relationship and recent results concerning the variance of the response of a random system.

Langley, R.S. JASA submitted, 2006