Random Matrix Models for Structural Dynamics

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Stochastic structural dynamics

- The equation of motion:

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{p}(t)$$

- Due to the presence of uncertainty $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ become random matrices.
- The main objectives are:
  - to quantify uncertainties in the system matrices
  - to predict the variability in the response vector $\mathbf{x}$

Current Methods

Three different approaches are currently available

- Low frequency : Stochastic Finite Element Method (SFEM) - considers parametric uncertainties in details
- High frequency : Statistical Energy Analysis (SEA) - do not consider parametric uncertainties in details
- Mid-frequency : Hybrid method - ‘combination’ of the above two

Random Matrix Method (RMM)

- The objective: To have an unified method which will work across the frequency range.
- The methodology:
  - Derive the matrix variate probability density functions of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$
  - Propagate the uncertainty (using Monte Carlo simulation or analytical methods) to obtain the response statistics (or pdf)

Outline of the presentation

In what follows next, I will discuss:

- Introduction to Matrix variate distributions
- Maximum entropy distribution
- Optimal Wishart distribution
- Numerical examples
- Open problems & discussions

Matrix variate distributions

- The probability density function of a random matrix can be defined in a manner similar to that of a random variable.
- If $\mathbf{A}$ is an $n \times m$ real random matrix, the matrix variate probability density function of $\mathbf{A} \in \mathbb{R}_{n,m}$, denoted as $p_{\mathbf{A}}(\mathbf{A})$, is a mapping from the space of $n \times m$ real matrices to the real line, i.e., $p_{\mathbf{A}}(\mathbf{A}) : \mathbb{R}_{n,m} \to \mathbb{R}$.

Gaussian random matrix

The random matrix $\mathbf{X} \in \mathbb{R}_{n,p}$ is said to have a matrix variate Gaussian distribution with mean matrix $\mathbf{M} \in \mathbb{R}_{n,p}$ and covariance matrix $\mathbf{\Sigma} \otimes \mathbf{\Psi}$, where $\mathbf{\Sigma} \in \mathbb{R}_n^+$ and $\mathbf{\Psi} \in \mathbb{R}_p^+$ provided the pdf of $\mathbf{X}$ is given by

$$p_{\mathbf{X}}(\mathbf{X}) = (2\pi)^{-np/2} |\mathbf{\Sigma}|^{-p/2} |\mathbf{\Psi}|^{-n/2} \\ \operatorname{etr}\left\{-\frac{1}{2}\mathbf{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})^T\right\} \quad (1)$$

This distribution is usually denoted as $\mathbf{X} \sim N_{n,p}(\mathbf{M}, \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Gaussian orthogonal ensembles

A random matrix $\mathbf{H} \in \mathbb{R}_{n,n}$ belongs to the Gaussian orthogonal ensemble (GOE) provided its pdf of is given by

$$p_{\mathbf{H}}(\mathbf{H}) = \exp\left(-\theta_2 \mathrm{Trace}\left(\mathbf{H}^2\right) + \theta_1 \mathrm{Trace}\left(\mathbf{H}\right) + \theta_0\right)$$

where θ_2 is real and positive and θ_1 and θ_0 are real.

This is a good model for high-frequency vibration problems.

Wishart matrix

An $n \times n$ random symmetric positive definite matrix $\mathbf{S}$ is said to have a Wishart distribution with parameters $p \geq n$ and $\boldsymbol{\Sigma} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{S}}\left(\mathbf{S}\right) = \left\{ 2^{\frac{1}{2}np}\, \Gamma_n \left(\frac{1}{2}p\right) \left|\boldsymbol{\Sigma}\right|^{\frac{1}{2}p} \right\}^{-1} \left|\mathbf{S}\right|^{\frac{1}{2}(p-n-1)} \mathrm{etr}\left\{ -\frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{S} \right\} \tag{2}$$

This distribution is usually denoted as $\mathbf{S} \sim W_n(p, \boldsymbol{\Sigma})$.

Note: If $p = n + 1$, then the matrix is non-negative definite.

Matrix variate Gamma distribution

An $n \times n$ random symmetric positive definite matrix $\mathbf{W}$ is said to have a matrix variate gamma distribution with parameters a and $\boldsymbol{\Psi} \in \mathbb{R}_n^+$, if its pdf is given by

$$p_{\mathbf{W}}(\mathbf{W}) = \left\{\Gamma_n(a) \left|\boldsymbol{\Psi}\right|^{-a}\right\}^{-1} \left|\mathbf{W}\right|^{a-\frac{1}{2}(n+1)} \operatorname{etr}\left\{-\boldsymbol{\Psi}\mathbf{W}\right\}; \quad \Re(a) > (n-1)/2 \quad (3)$$

This distribution is usually denoted as $\mathbf{W} \sim G_n(a, \boldsymbol{\Psi})$. Here the multivariate gamma function:

$$\Gamma_n(a) = \pi^{\frac{1}{4}n(n-1)} \prod_{k=1}^{n} \Gamma\left[a - \frac{1}{2}(k-1)\right]; \text{ for } \Re(a) > (n-1)/2 \quad (4)$$

Distribution of the system matrices

The distribution of the random system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ should be such that they are

- symmetric
- positive-definite, and
- the moments (at least first two) of the inverse of the dynamic stiffness matrix $\mathbf{D}(\omega) = -\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}$ should exist $\forall\,\omega$

Distribution of the system matrices

- The exact application of the last constraint requires the derivation of the joint probability density function of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$, which is quite difficult to obtain.
- We consider a simpler problem where it is required that the inverse moments of each of the system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ must exist.
- Provided the system is damped, this will guarantee the existence of the moments of the frequency response function matrix.

Maximum Entropy Distribution

Suppose that the mean values of $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are given by $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$ respectively. Using the notation $\mathbf{G}$ (which stands for any one the system matrices) the matrix variate density function of $\mathbf{G} \in \mathbb{R}_n^+$ is given by $p_{\mathbf{G}}(\mathbf{G}) : \mathbb{R}_n^+ \rightarrow \mathbb{R}$. We have the following constrains to obtain $p_{\mathbf{G}}(\mathbf{G})$:

$$\int_{\mathbf{G}>0} p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = 1 \quad \text{(normalization)} \tag{5}$$

and $$\int_{\mathbf{G}>0} \mathbf{G}\, p_{\mathbf{G}}(\mathbf{G})\, d\mathbf{G} = \overline{\mathbf{G}} \quad \text{(the mean matrix)} \tag{6}$$

Further constraints

- Suppose the inverse moments (say up to order ν) of the system matrix exist. This implies that $\mathrm{E}\left[\left\|\mathbf{G}^{-1}\right\|_{\mathrm{F}}^{\nu}\right]$ should be finite. Here the Frobenius norm of matrix $\mathbf{A}$ is given by $\|\mathbf{A}\|_{\mathrm{F}} = \left(\mathrm{Trace}\left(\mathbf{A}\mathbf{A}^{T}\right)\right)^{1/2}$.
- Taking the logarithm for convenience, the condition for the existence of the inverse moments can be expresses by

$$\mathrm{E}\left[\ln|\mathbf{G}|^{-\nu}\right] < \infty$$

MEnt Distribution - 1

The Lagrangian becomes:

$$\mathcal{L}\left(p_{\mathbf{G}}\right) = -\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right) \ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\} \, d\mathbf{G} + \left(\lambda_0 - 1\right)\left(\int_{\mathbf{G}>0} p_{\mathbf{G}}\left(\mathbf{G}\right) \, d\mathbf{G} - 1\right) - \nu \int_{\mathbf{G}>0} \ln|\mathbf{G}| \, p_{\mathbf{G}} \, d\mathbf{G} + \operatorname{Trace}\left(\mathbf{\Lambda}_1 \left[\int_{\mathbf{G}>0} \mathbf{G} \, p_{\mathbf{G}}\left(\mathbf{G}\right) \, d\mathbf{G} - \overline{\mathbf{G}}\right]\right) \tag{7}$$

Note: ν cannot be obtained uniquely!

MEnt Distribution - 2

Using the calculus of variation

$$\frac{\partial \mathcal{L}\left(p_{\mathbf{G}}\right)}{\partial p_{\mathbf{G}}} = 0$$

or $-\ln\left\{p_{\mathbf{G}}\left(\mathbf{G}\right)\right\} = \lambda_0 + \mathrm{Trace}\left(\mathbf{\Lambda}_1 \mathbf{G}\right) - \ln\left|\mathbf{G}\right|^{\nu}$

or $p_{\mathbf{G}}\left(\mathbf{G}\right) = \exp\left\{-\lambda_0\right\}\left|\mathbf{G}\right|^{\nu} \mathrm{etr}\left\{-\mathbf{\Lambda}_1 \mathbf{G}\right\}$

MEnt Distribution - 3

Using the matrix variate Laplace transform **($\mathbf{T} \in \mathbb{R}_{n,n}, \mathbf{S} \in \mathbb{C}_{n,n}, a > (n+1)/2$)**

$$\int_{\mathbf{T}>0} \mathrm{etr}\left\{-\mathbf{S}\mathbf{T}\right\} |\mathbf{T}|^{a-(n+1)/2} \, d\mathbf{T} = \Gamma_n(a) \, |\mathbf{S}|^{-a}$$

and substituting $p_{\mathbf{G}}(\mathbf{G})$ into the constraint equations it can be shown that

$$p_{\mathbf{G}}(\mathbf{G}) = r^{-nr} \left\{\Gamma_n(r)\right\}^{-1} \left|\overline{\mathbf{G}}\right|^{-r} |\mathbf{G}|^{\nu} \, \mathrm{etr}\left\{-r\overline{\mathbf{G}}^{-1}\mathbf{G}\right\} \tag{8}$$

where $r = \nu + (n+1)/2$.

MEnt Distribution - 4

Comparing it with the Wishart distribution we have:

Theorem 1. *If ν-th order inverse-moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of $\mathbf{G}$ is available, say $\overline{\mathbf{G}}$, then the maximum-entropy pdf of $\mathbf{G}$ follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and $\mathbf{\Sigma} = \overline{\mathbf{G}}/(2\nu + n + 1)$, that is $\mathbf{G} \sim W_n \left(2\nu + n + 1, \overline{\mathbf{G}}/(2\nu + n + 1)\right)$.*

Properties of the Distribution

- Covariance tensor of $\mathbf{G}$:

$$\mathrm{cov}\left(G_{ij}, G_{kl}\right) = \frac{1}{2\nu + n + 1}\left(\overline{G}_{ik}\overline{G}_{jl} + \overline{G}_{il}\overline{G}_{jk}\right)$$

- Normalized standard deviation matrix

$$\delta^2_{\mathbf{G}} = \frac{\mathrm{E}\left[\|\mathbf{G} - \mathrm{E}\left[\mathbf{G}\right]\|^2_{\mathrm{F}}\right]}{\|\mathrm{E}\left[\mathbf{G}\right]\|^2_{\mathrm{F}}} = \frac{1}{2\nu + n + 1}\left\{1 + \frac{\left\{\mathrm{Trace}\left(\overline{\mathbf{G}}\right)\right\}^2}{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)}\right\}$$

- $\delta^2_{\mathbf{G}} \leq \dfrac{1+n}{2\nu+n+1}$ and $\nu \uparrow \Rightarrow \delta^2_{\mathbf{G}} \downarrow$.

Distribution of the inverse - 1

- If $\mathbf{G}$ is $W_n(p, \boldsymbol{\Sigma})$ then $\mathbf{V} = \mathbf{G}^{-1}$ has the inverted Wishart distribution:

$$P_{\mathbf{V}}(\mathbf{V}) = \frac{2^{m-n-1}n/2 \left|\boldsymbol{\Psi}\right|^{m-n-1}/2}{\Gamma_n[(m-n-1)/2] \left|\mathbf{V}\right|^{m/2}} \mathrm{etr}\left\{-\frac{1}{2}\mathbf{V}^{-1}\boldsymbol{\Psi}\right\}$$

where $m = n + p + 1$ and $\boldsymbol{\Psi} = \boldsymbol{\Sigma}^{-1}$ (recall that $p = 2\nu + n + 1$ and $\boldsymbol{\Sigma} = \overline{\mathbf{G}}/p$)

Distribution of the inverse - 2

- Mean: $\mathrm{E}\left[\mathbf{G}^{-1}\right] = \dfrac{p\overline{\mathbf{G}}^{-1}}{p-n-1}$
- $\mathrm{cov}\left(G_{ij}^{-1}, G_{kl}^{-1}\right) =$
$$\frac{\left(2\nu+n+1)(\nu^{-1}\overline{G}_{ij}^{-1}\overline{G}_{kl}^{-1}+\overline{G}_{ik}^{-1}\overline{G}_{jl}^{-1}+\overline{G}^{-1}il\overline{G}_{kj}^{-1}\right)}{2\nu(2\nu+1)(2\nu-2)}$$

Distribution of the inverse - 3

- Suppose $n = 101$ & $\nu = 2$. So $p = 2\nu + n + 1 = 106$ and $p - n - 1 = 4$. Therefore, $\mathrm{E}\left[\mathbf{G}\right] = \overline{\mathbf{G}}$ and $\mathrm{E}\left[\mathbf{G}^{-1}\right] = \frac{106}{4}\overline{\mathbf{G}}^{-1} = 26.5\overline{\mathbf{G}}^{-1}$!!!!!!!!!!
- From a practical point of view we do not expect them to be so far apart!
- One way to reduce the gap is to increase p. But this implies the reduction of variance.

Optimal Wishart Distribution - 1

- My argument: The distribution of $\mathbf{G}$ must be such that $\mathrm{E}\left[\mathbf{G}\right]$ and $\mathrm{E}\left[\mathbf{G}^{-1}\right]$ should be closest to $\overline{\mathbf{G}}$ and $\overline{\mathbf{G}}^{-1}$ respectively.
- Suppose $\mathbf{G} \sim W_n\left(n+1+\theta, \overline{\mathbf{G}}/\alpha\right)$. We need to find α such that the above condition is satisfied.
- Therefore, define (and subsequently minimize) 'normalized errors':

$$\varepsilon_1 = \left\|\overline{\mathbf{G}} - \mathrm{E}\left[\mathbf{G}\right]\right\|_{\mathrm{F}} / \left\|\overline{\mathbf{G}}\right\|_{\mathrm{F}}$$

$$\varepsilon_2 = \left\|\overline{\mathbf{G}}^{-1} - \mathrm{E}\left[\mathbf{G}^{-1}\right]\right\|_{\mathrm{F}} / \left\|\overline{\mathbf{G}}^{-1}\right\|_{\mathrm{F}}$$

Optimal Wishart Distribution - 2

Because $\mathbf{G} \sim W_n\left(n+1+\theta, \overline{\mathbf{G}}/\alpha\right)$ we have

$$\mathrm{E}\left[\mathbf{G}\right] = \frac{n+1+\theta}{\alpha}\overline{\mathbf{G}}$$

$$\text{and} \quad \mathrm{E}\left[\mathbf{G}^{-1}\right] = \frac{\alpha}{\theta}\overline{\mathbf{G}}^{-1}$$

We define the objective function to be minimized as $\chi^2 = {\boldsymbol{\varepsilon}_1}^2 + {\boldsymbol{\varepsilon}_2}^2 = \left(1 - \frac{n+1+\theta}{\alpha}\right)^2 + \left(1 - \frac{\alpha}{\theta}\right)^2$

Optimal Wishart Distribution - 3

The optimal value of α can be obtained as by setting $\frac{\partial \chi^2}{\partial \alpha} = 0$ or
$\alpha^4 - \alpha^3\theta - \theta^4 + (-2\,n + \alpha - 2)\,\theta^3 + \big((n+1)\,\alpha - n^2 - 2\,n - 1\big)\,\theta^2 = 0.$
The only feasible value of α is

$$\alpha = \sqrt{\theta(n + 1 + \theta)}$$

Optimal Wishart Distribution - 4

From this discussion we have the following:

Theorem 2. *If ν-th order inverse-moment of a system matrix $\mathbf{G} \equiv \{\mathbf{M}, \mathbf{C}, \mathbf{K}\}$ exists and only the mean of $\mathbf{G}$ is available, say $\overline{\mathbf{G}}$, then the unbiased distribution of $\mathbf{G}$ follows the Wishart distribution with parameters $p = (2\nu + n + 1)$ and $\boldsymbol{\Sigma} = \overline{\mathbf{G}} / \sqrt{2\nu(2\nu + n + 1)}$, that is*

$$\mathbf{G} \sim W_n \left(2\nu + n + 1, \overline{\mathbf{G}} / \sqrt{2\nu(2\nu + n + 1)} \right).$$

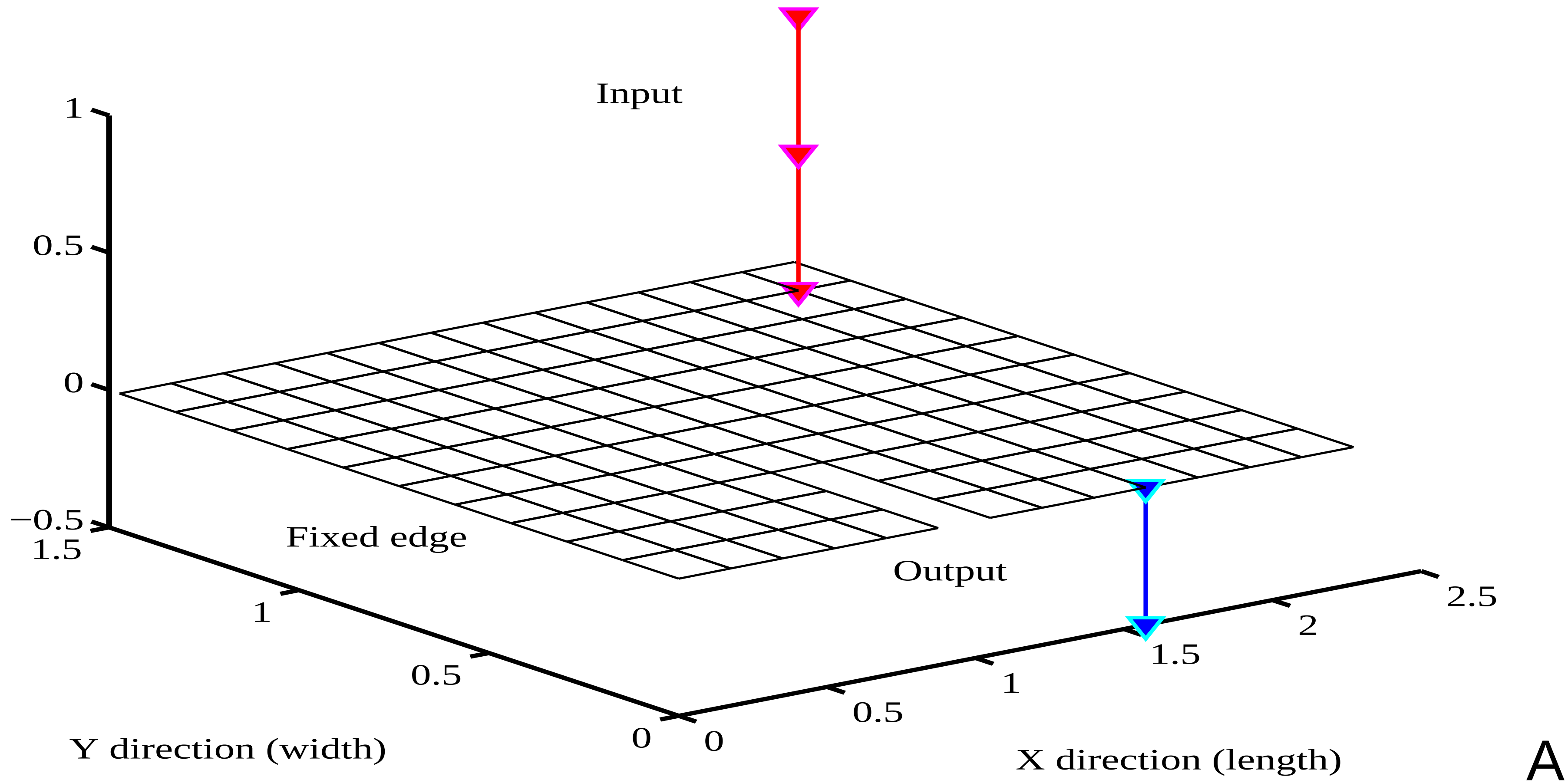
Optimal Wishart Distribution - 5

- Again consider $n = 100$ and $\nu = 2$, so that $\theta = 2\nu = 4$.
- In the previous approach $\alpha = 2\nu + n + 1 = 105$. For the optimal distribution, $\alpha = \sqrt{\theta(\theta + n + 1)} = 2\sqrt{105} = 20.49$.
- We have $\mathrm{E}\left[\mathbf{G}\right] = \frac{105}{2\sqrt{105}}\overline{\mathbf{G}} = 5.12\overline{\mathbf{G}}$ and $\mathrm{E}\left[\mathbf{G}^{-1}\right] = \frac{2\sqrt{105}}{4}\overline{\mathbf{G}}^{-1} = 5.12\overline{\mathbf{G}}^{-1}$.
- The overall normalized difference for the previous case is $\chi^2 = 0 + (1 - 105/4)^2 = 637.56$. The same for the optimal distribution is $\chi^2 = 2(1 - \sqrt{105}/2)^2 = 34.01$, which is considerable smaller compared to the non-optimal distribution.

Simulation Algorithm

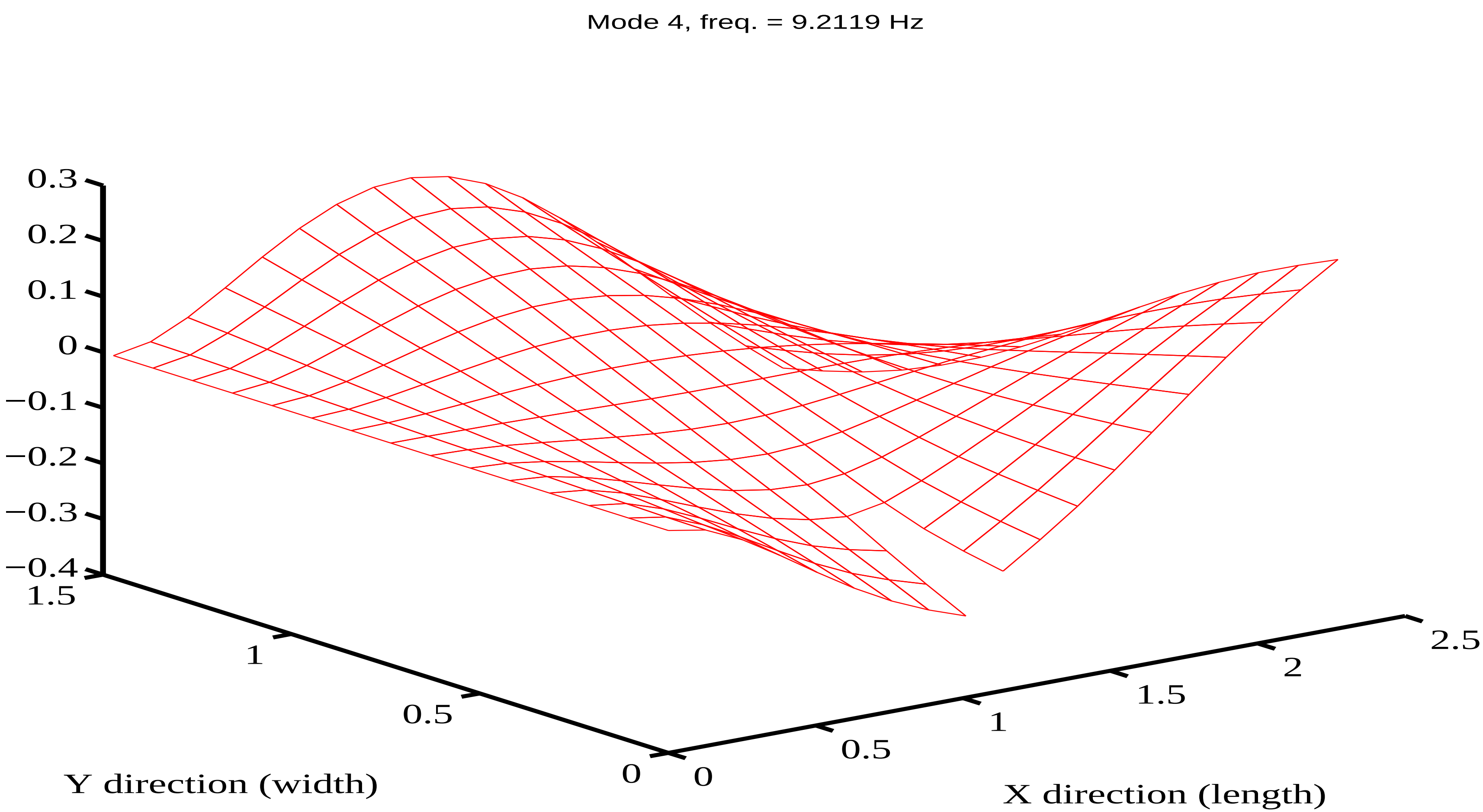
- Obtain $\theta = \frac{1}{\delta_{\mathbf{G}}^2}\left\{1 + \frac{\{\mathrm{Trace}\left(\overline{\mathbf{G}}\right)\}^2}{\mathrm{Trace}\left(\overline{\mathbf{G}}^2\right)}\right\} - (n+1)$
- If $\theta < 4$, then select $\theta = 4$.
- Calculate $\alpha = \sqrt{\theta(n+1+\theta)}$
- Generate samples of $\mathbf{G} \sim W_n\left(n+1+\theta, \overline{\mathbf{G}}/\alpha\right)$ (MATLAB® command `wishrnd` can be used to generate the samples)
- Repeat the above steps for all system matrices and solve for every samples

Example: A cantilever Plate



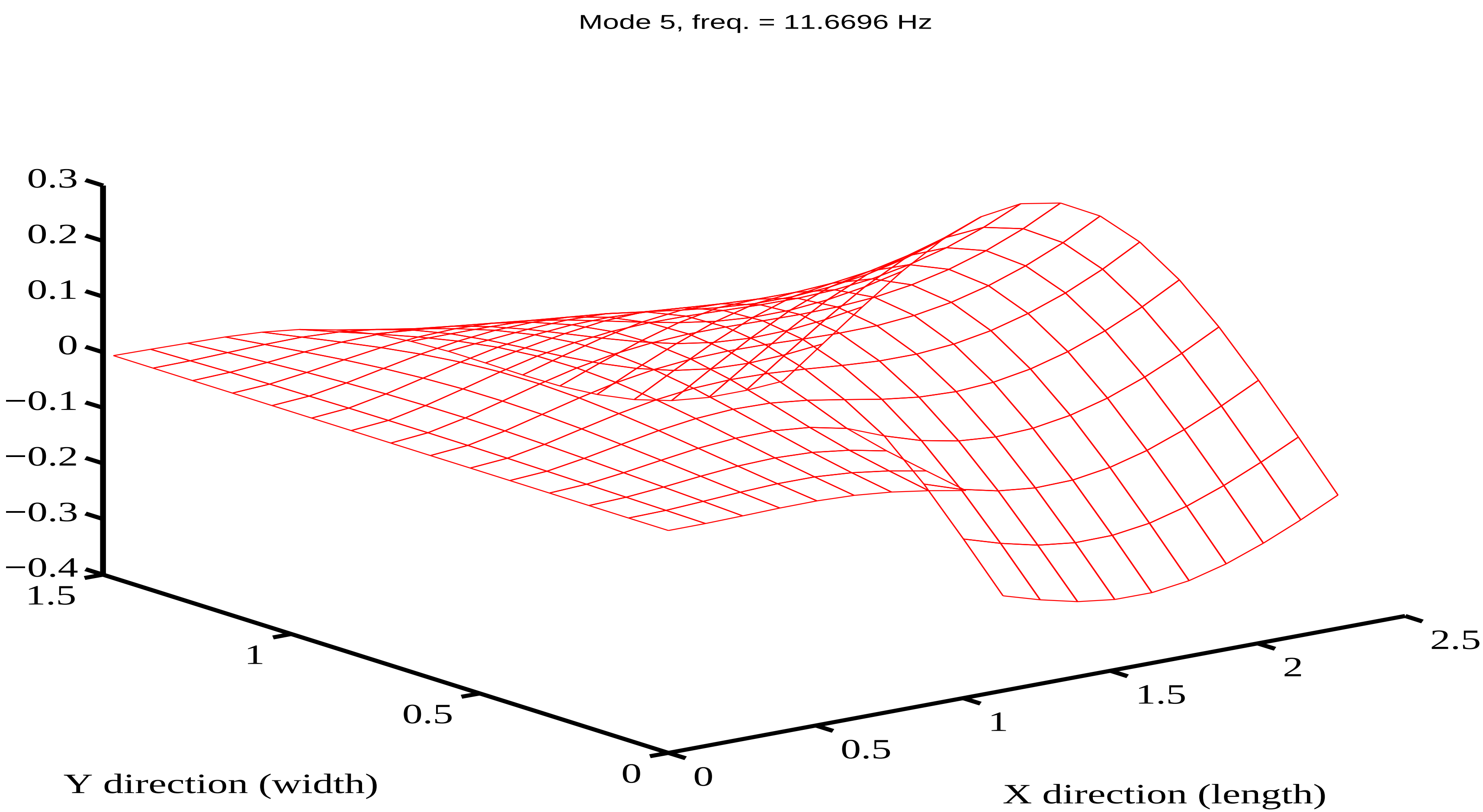
Cantilever plate with a slot: $\mu = 0.3$, $\rho = 8000$ kg/m^3, $t = 5$mm, $L_x = 2.27$m, $L_y = 1.47$m

Plate Mode 4



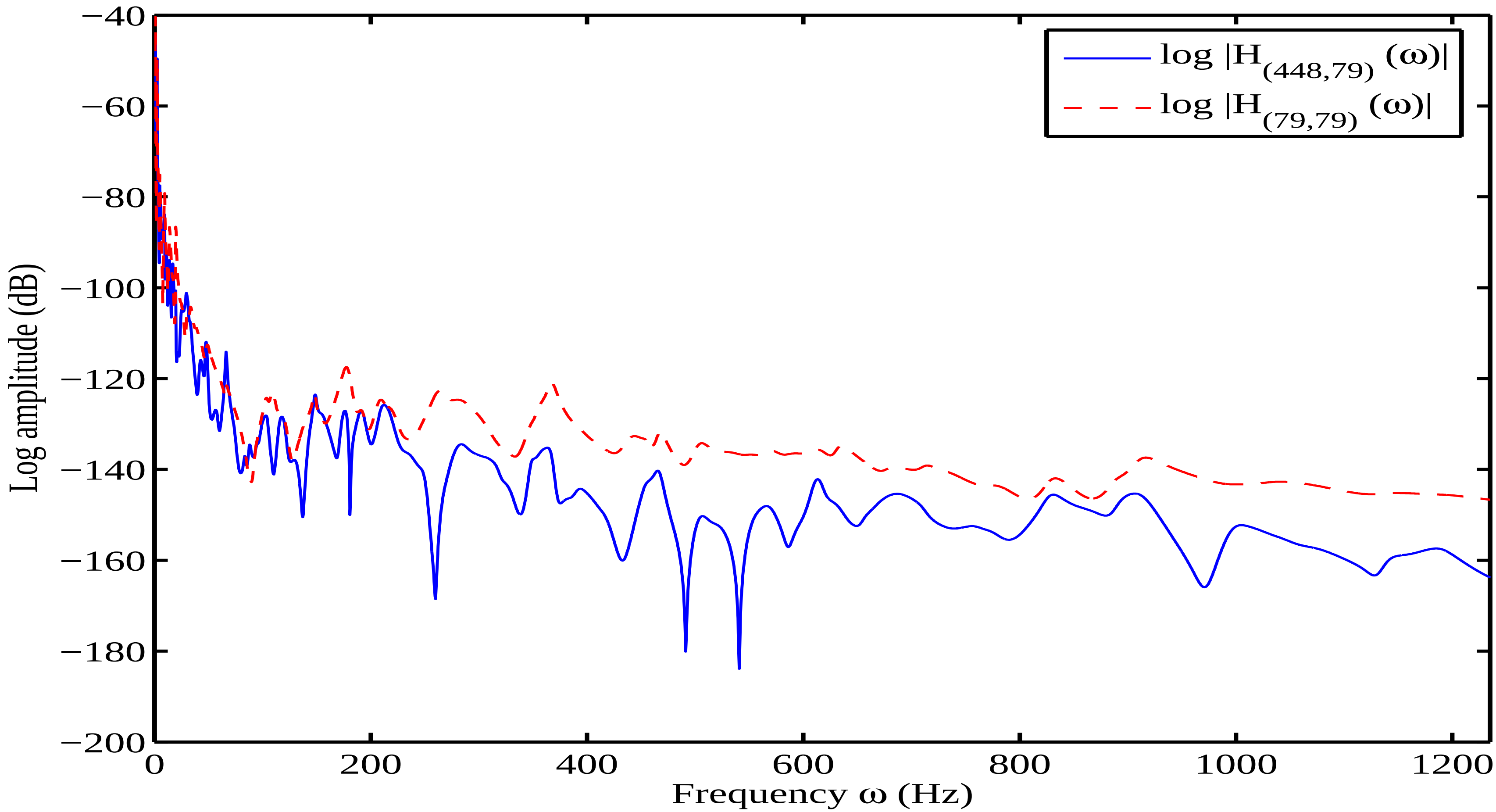
Fourth Mode shape

Plate Mode 5



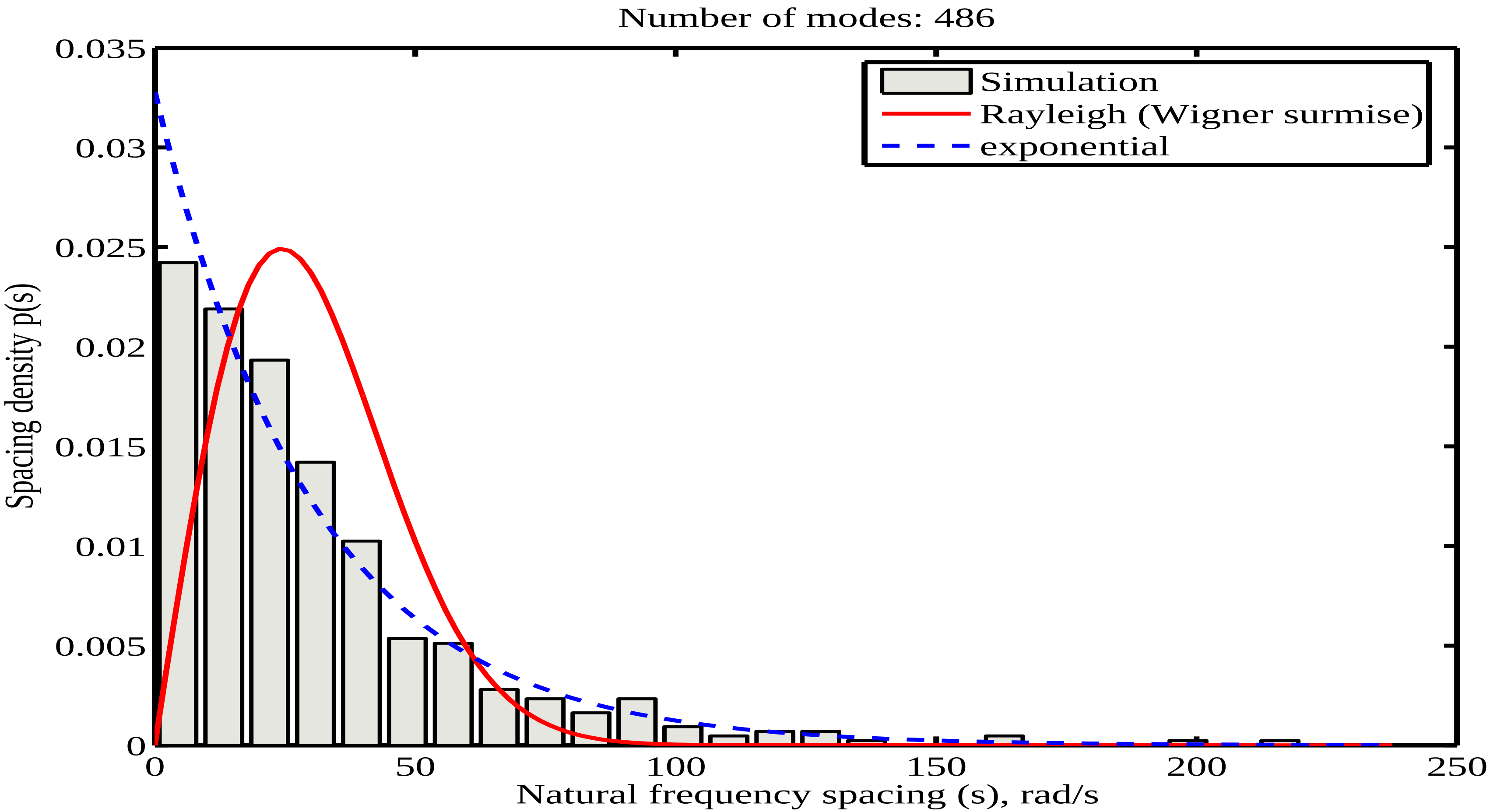
Fifth Mode shape

Deterministic FRF



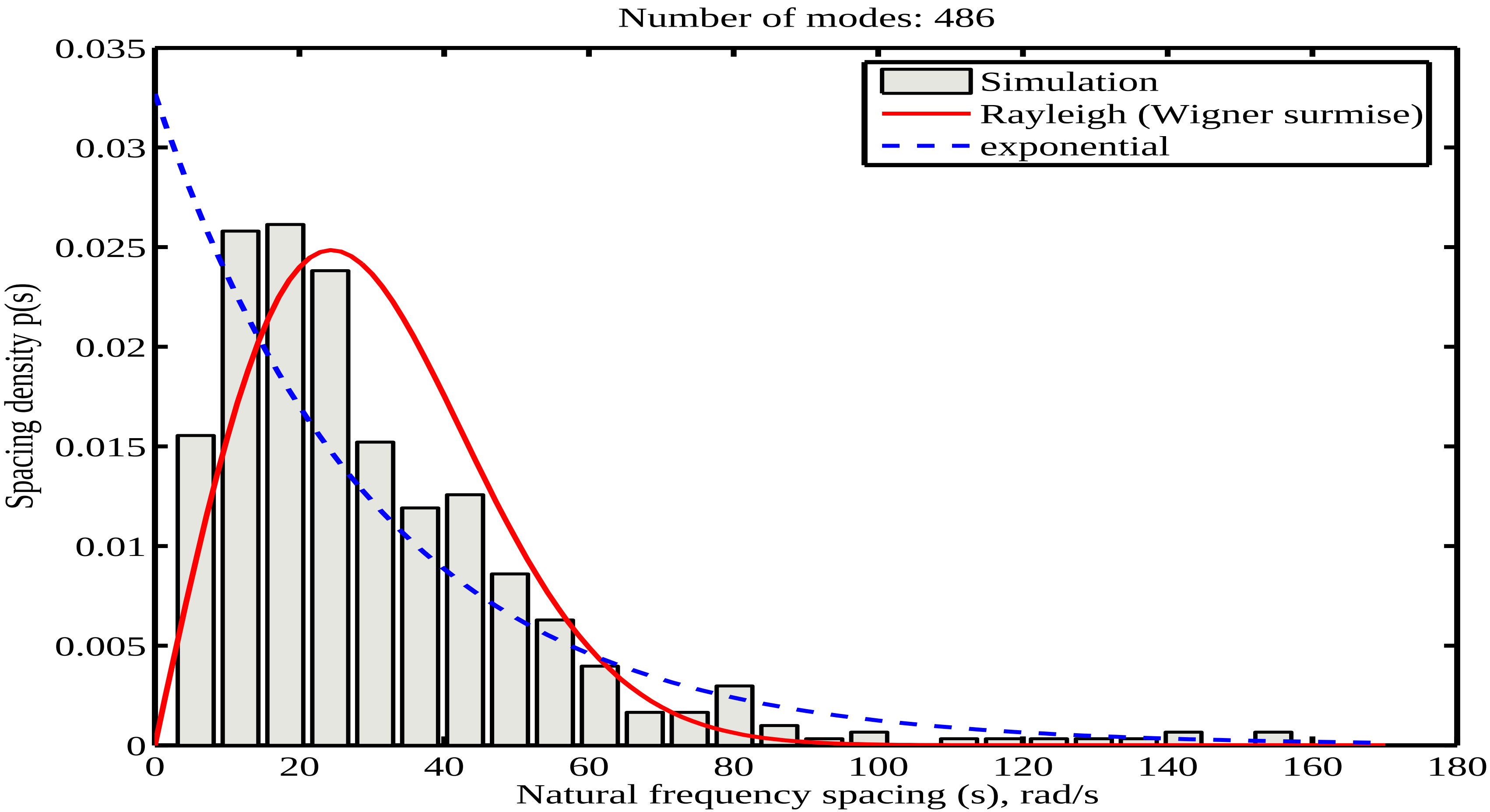
FRF of the deterministic plate

Frequency Spacing



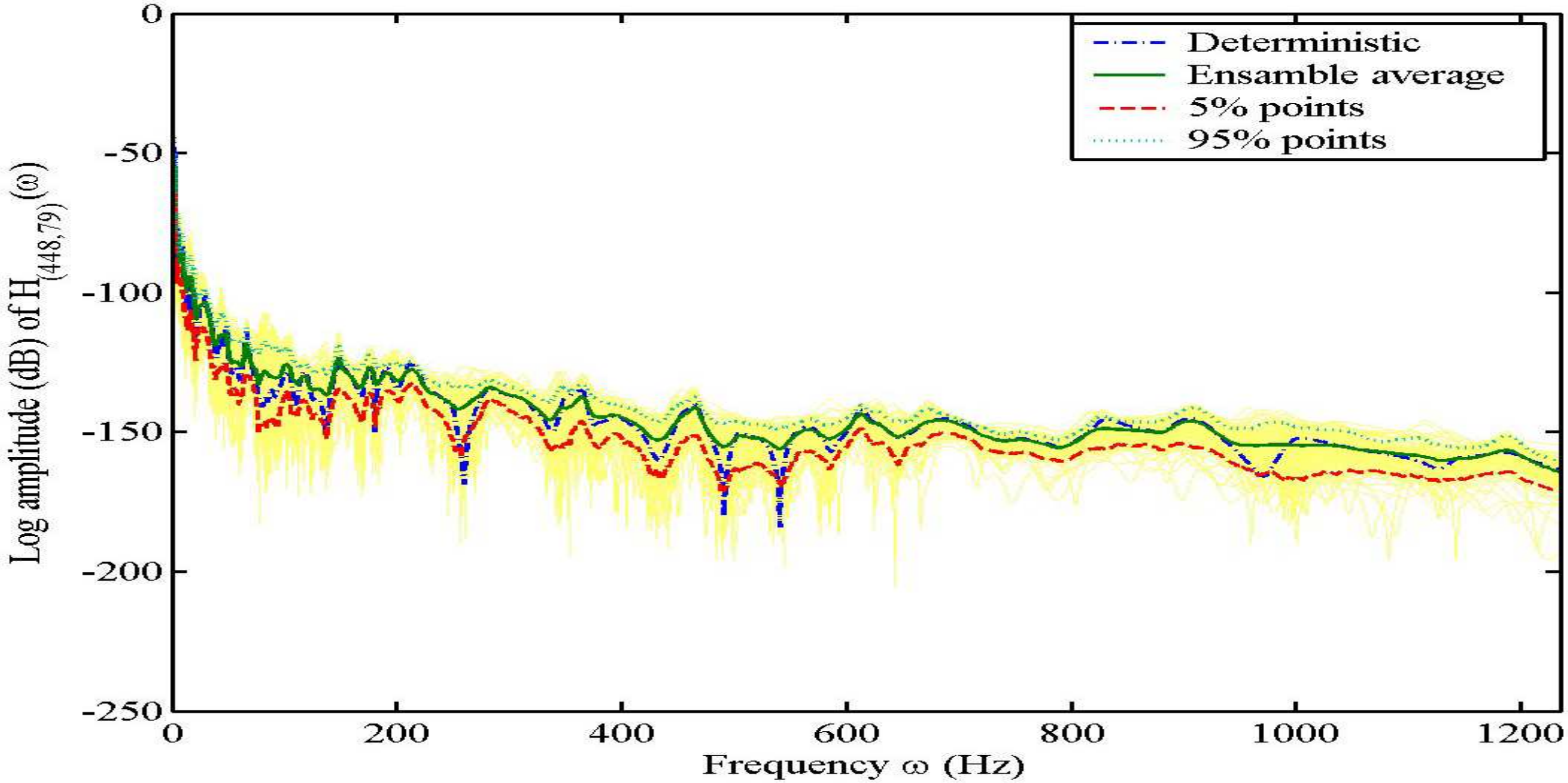
Natural frequency spacing distribution (without slot)

Frequency Spacing



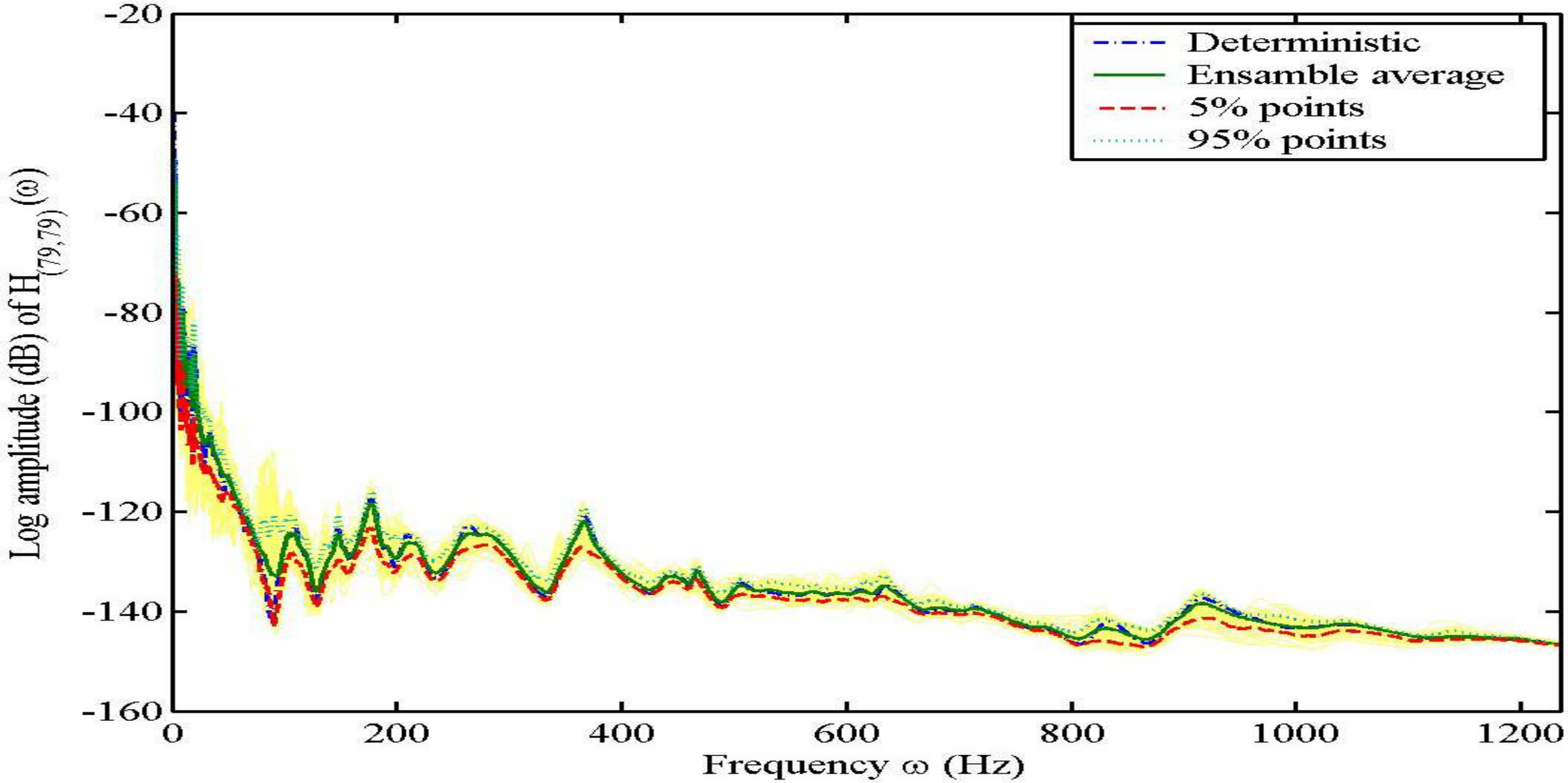
Natural frequency spacing distribution (with slot)

Random FRF - 1



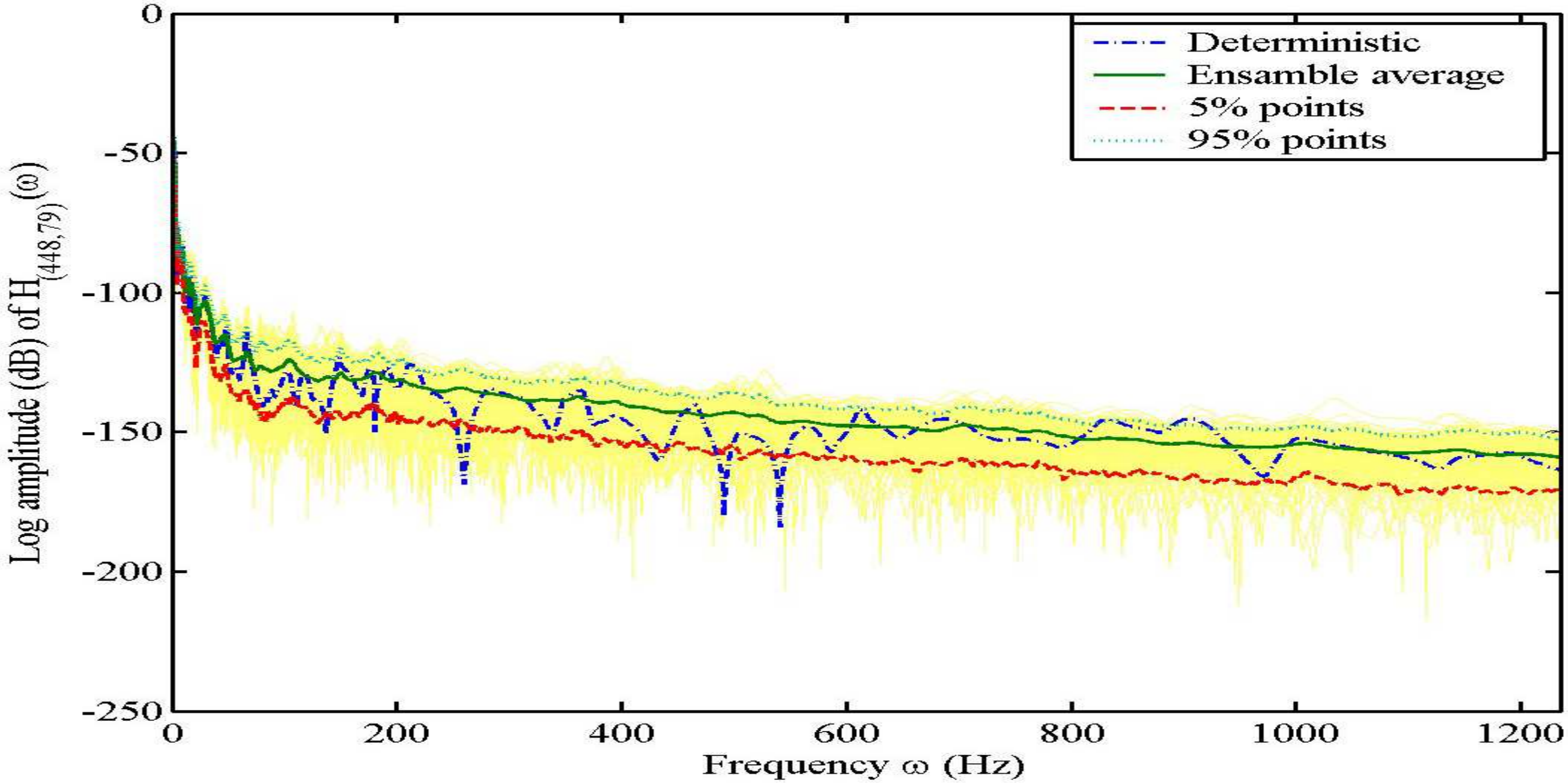
Direct finite-element MCS of the amplitude of the cross-FRF of the plate with randomly placed masses; 30 masses, each weighting 0.5% of the total mass of the plate are simulated.

Random FRF - 2



Direct finite-element MCS of the amplitude of the driving-point FRF of the plate with randomly placed masses; 30 masses, each weighting 0.5% of the total mass of the plate are simulated.

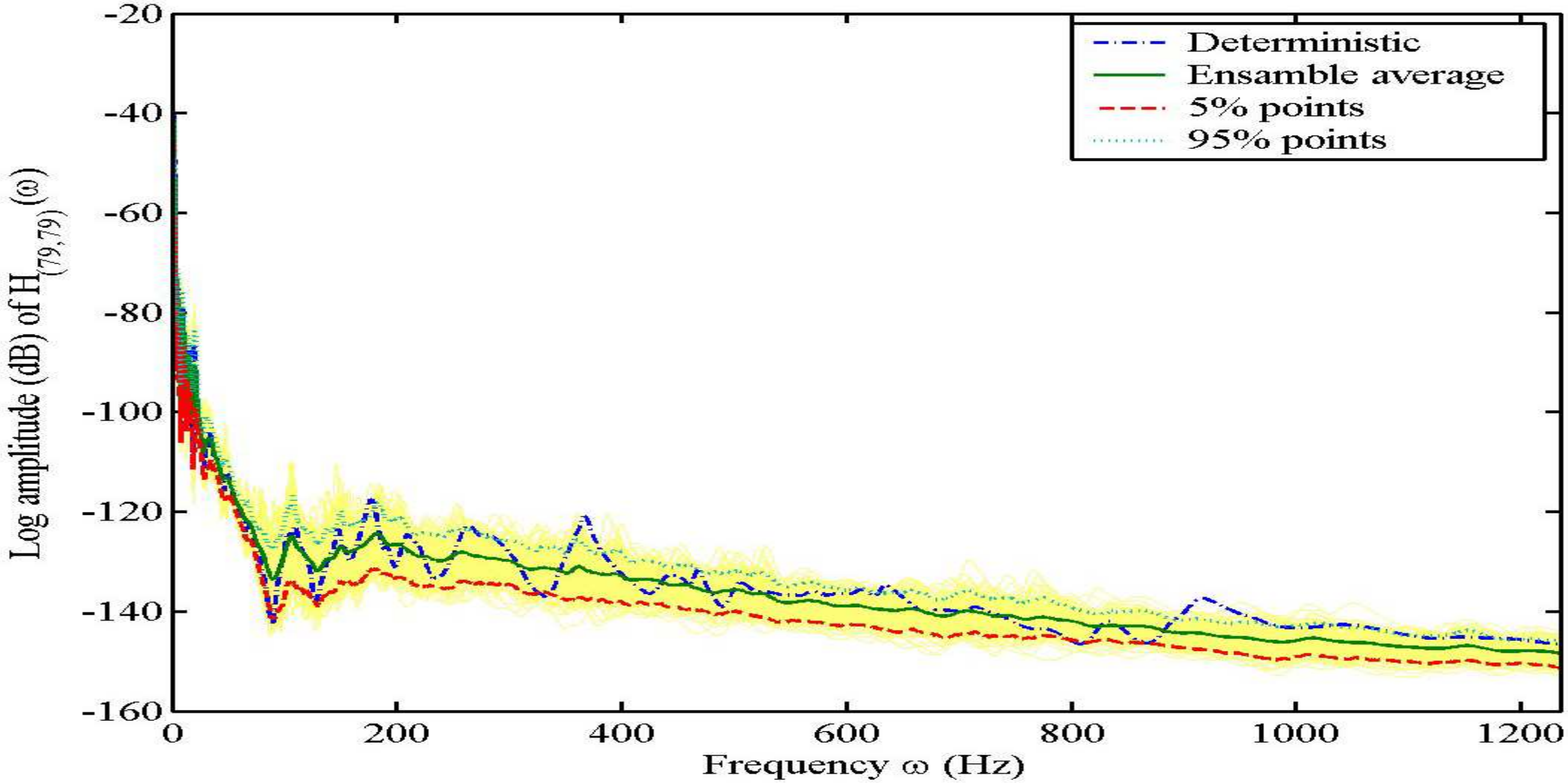
Wishart FRF - 1



MCS of the amplitude of the cross-FRF of the plate using optimal Wishart mass matrix,

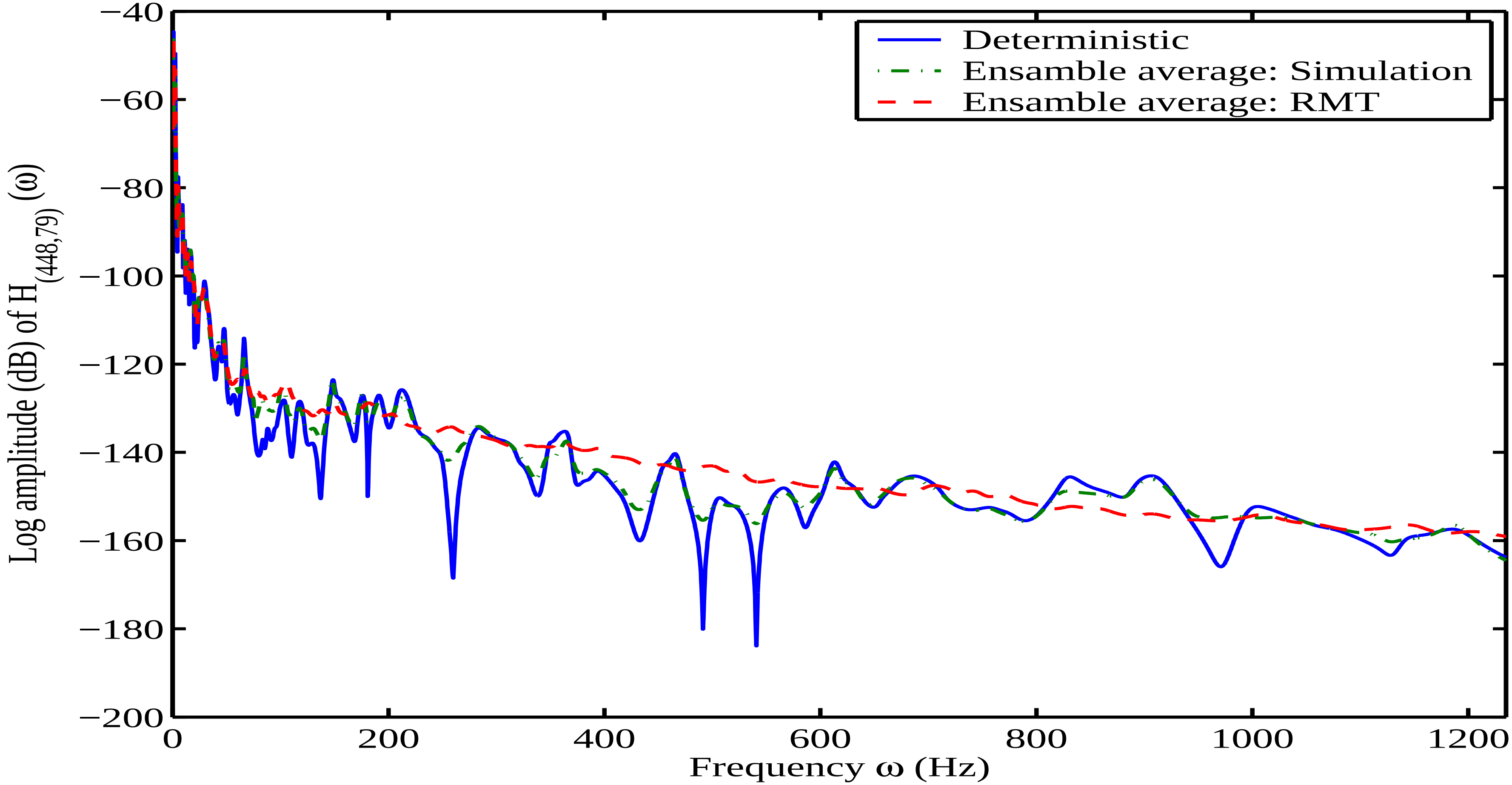
$n = 429$, $\delta_M = 2.0449$.

Wishart FRF - 2



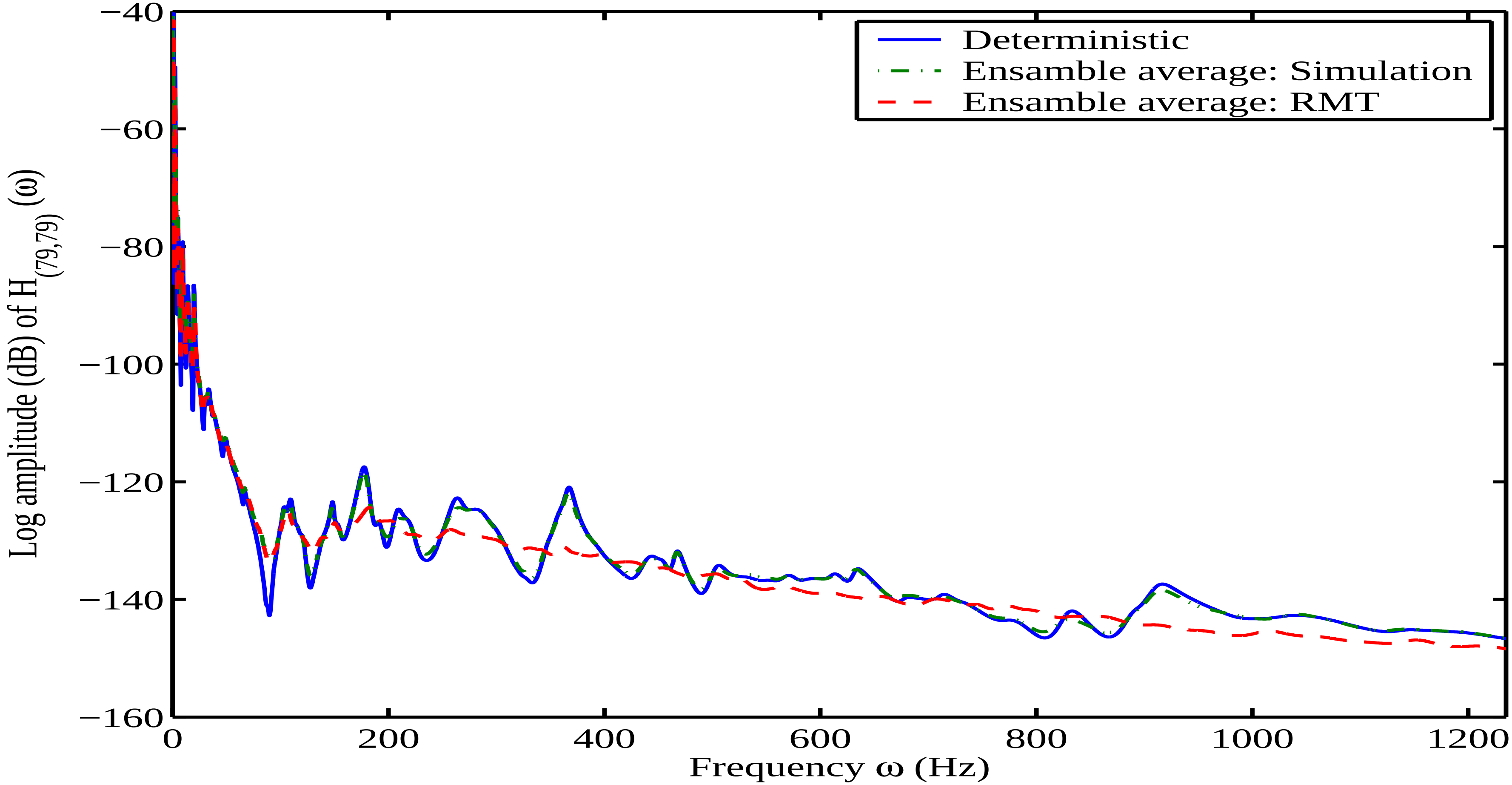
MCS of the amplitude of the driving-point-FRF of the plate using optimal Wishart mass matrix, $n = 429$, $\delta_M = 2.0449$.

Comparison of Mean - 1



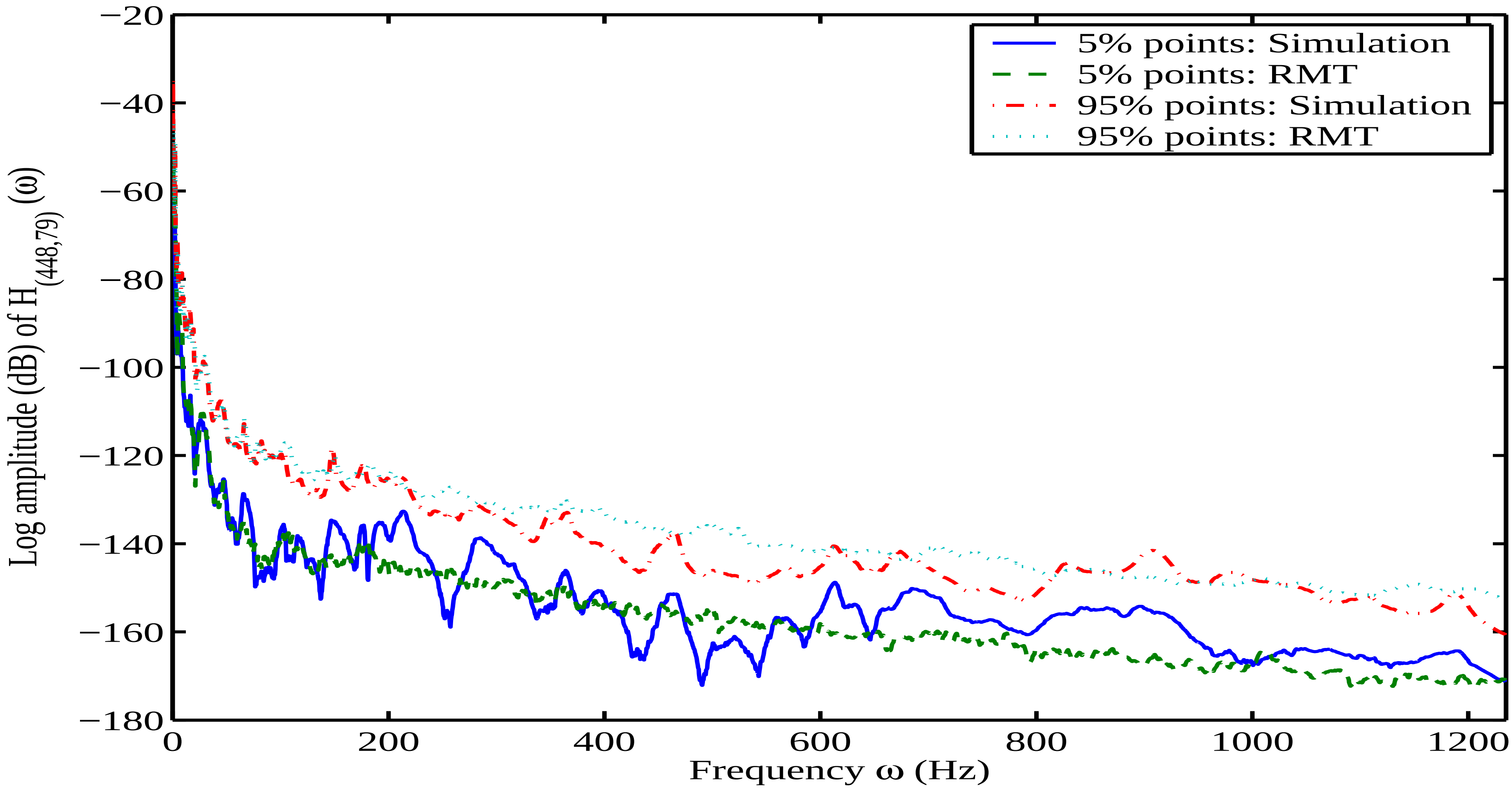
Comparison of the mean values of the amplitude of the cross-FRF.

Comparison of Mean - 2



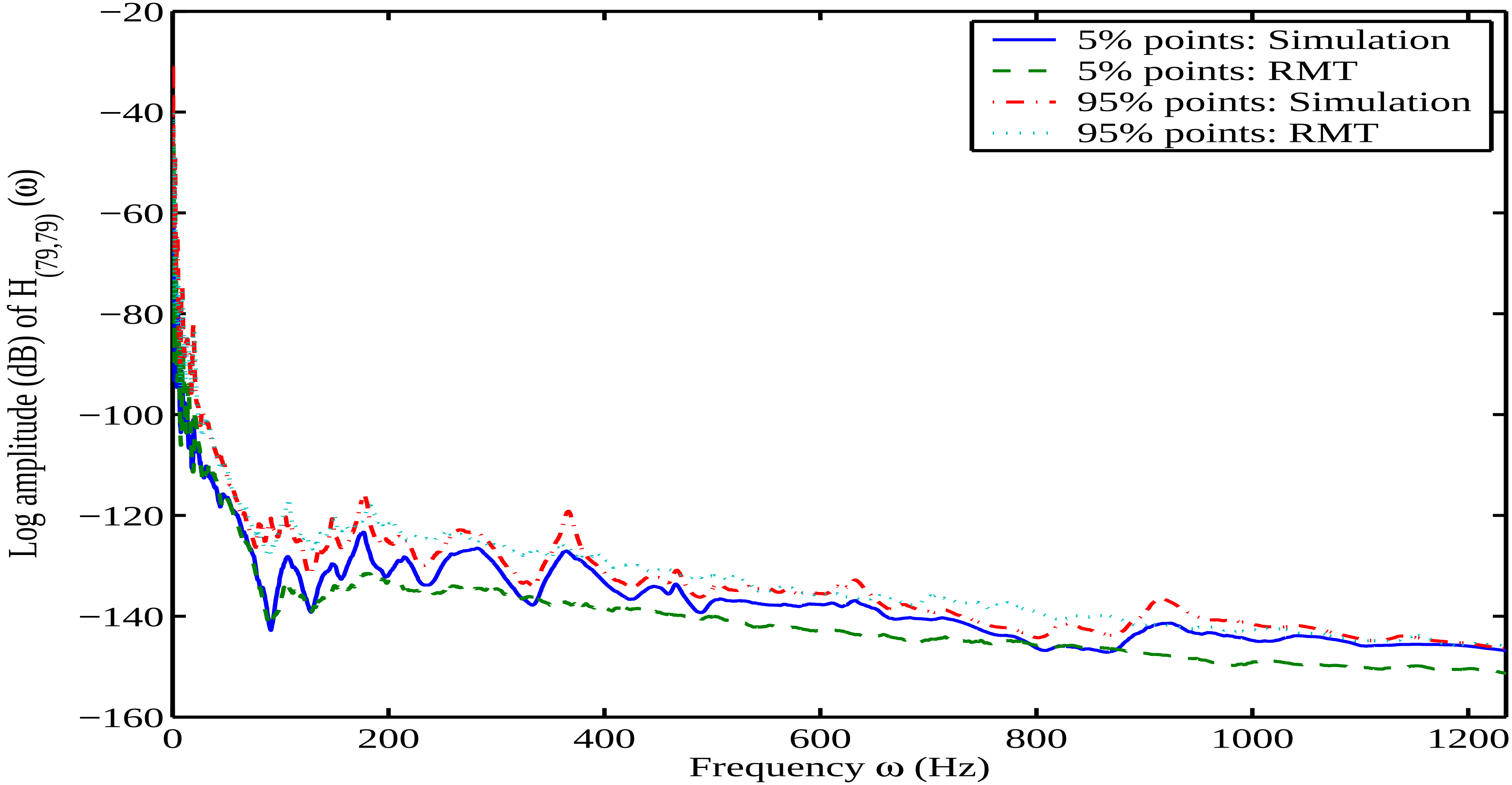
Comparison of the mean values of the amplitude of the driving-point-FRF.

Comparison of Variation - 1



Comparison of the 5% and 95% probability points of the amplitude of the cross-FRF.

Comparison of Variation - 2



Comparison of the 5% and 95% probability points of the amplitude of the driving-point-FRF.

Summary & conclusions

- Wishart matrices can used as the distribution for the system matrices in structural dynamics.
- The parameters of the distribution can be obtained by solving an optimisation problem

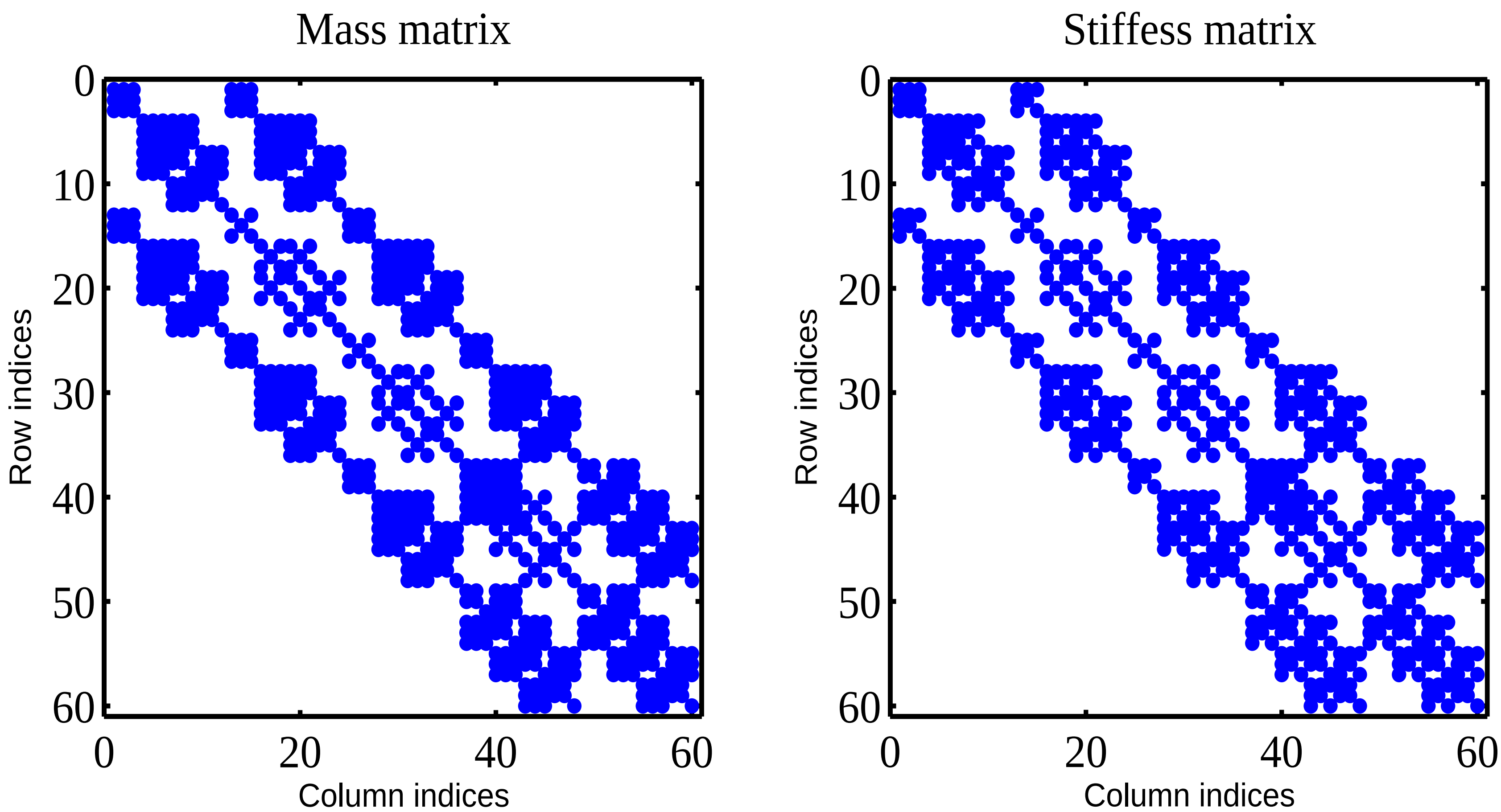
Next steps

- Numerical works (validation against??)
- Eigenvalues, eigenvector statistics and calculation of dynamic response.
- Distribution of the dynamic stiffness matrix (complex Wishart matrix?)
- Inversion of the dynamic stiffness matrix (FRFs)
- Distribution of $\mathbf{Y}(\omega) = \left[\mathbf{R}\mathbf{D}(\omega)^{-1}\mathbf{P}\right]$ where $\mathbf{P} \in \mathbb{C}_{n,r}$ and $\mathbf{R} \in \mathbb{R}_{p,n}$
- Cumulative distribution function of the response (reliability problem)

Open problems & discussions

- Is MEnT appropriate here?
- $\overline{\mathbf{G}}$ is just one ‘observation’ - not an ensemble mean.
- What happens if we know the covariance tensor of $\mathbf{G}$ (e.g., using Stochastic Finite element Method)?
- What if the zeros in $\mathbf{G}$ are not preserved?

Structure of the Matrices



Nonzero elements of the system matrices