Extremely strong convergence of eigenvalue-density of linear stochastic dynamical systems

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Abstract Eigenvalue problems play a crucial role in the stability and dynamics of engineering systems modeled using the linear mechanical theory. When uncertainties, either in the parameters or in the modelling, are considered, the eigenvalue problem becomes a random eigenvalue problem. Over the past half a century, random eigenvalue problems have received extensive attentions from the physicists, applied mathematicians and engineers. Within the context of civil, mechanical and aerospace engineering, significant work has been done on perturbation method based approaches in conjunction with the stochastic finite element method. The perturbation based methods work very well in the low frequency region which is often sufficient for many engineering applications. In the high frequency region however, which is necessary for some practical applications, these methods often fail to capture crucial physics, such as the veering and modal overlap. In this region one needs to consider the complete spectrum of the eigenvalues as opposed to the individual eigenvalues often considered in the low frequency applications. In this paper we consider the density of the eigenvalues of a discrete or discretised continuous system with uncertainty. It has been rigorously proved that the density of eigenvalues of random dynamical systems reaches a non-random limit for large systems. This fact has been demonstrated by numerical examples. The implications of this result for the response calculation of large stochastic structural dynamical systems have been highlighted.

1 Introduction

The predictions from high-resolution numerical models may sometimes exhibit significant differences with the results from physical experiments due to uncertainty. Such uncertainties include, but are not limited to (a) parameter uncertainty (e.g, uncertainty in geometric parameters, friction coefficient, strength of the materials involved); (b) model uncertainty (arising from the lack of scientific knowledge about the model which is *a priori* unknown); (c) experimental error (uncertain and unknown errors percolate into the model when they are calibrated against experimental results). When substantial statistical information exists, the theory of probability and stochastic processes offer a rich mathematical framework to represent such uncertainties. In a probabilistic setting, the data (parameter) uncertainty associated with the system parameters, such as the geometric properties and constitutive relations (i.e. Young's modulus, mass density, Poisson's ratio, damping coefficients), can be modeled as random variables or stochastic processes using the so-called parametric approach. These uncertainties can be quantified and propagated, for example, using the stochastic finite element method

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[5, 6, 15, 16, 20, 24, 30, 31, 34, 35, 40, 42, 43, 49]. Recently, the uncertainty due to modelling error has received attention as this is crucial for model validation [21–23]. The model uncertainty problem poses serious challenges as the parameters contributing to the modelling errors are not available *a priori* and therefore precludes the application of a parametric approach to address such issues. Model uncertainties do not explicitly depend on the system parameters. For example, there can be unquantified errors associated with the equation of motion (linear or non-linear), in the damping model (viscous or non-viscous [8, 9]), in the model of structural joints. The model uncertainty may be tackled by the so-called non-parametric method such as the statistical energy analysis [25–29, 44] random matrix approach [1–3, 7, 38, 46–48].

The equation of motion of a damped n-degree-of-freedom linear dynamic system can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{f}(t) \in \mathbb{R}^n$ is the forcing vector, $\mathbf{q}(t) \in \mathbb{R}^n$ is the response vector and $\mathbf{M} \in \mathbb{R}^{n\times n}$, $\mathbf{C} \in \mathbb{R}^{n\times n}$ and $\mathbf{K} \in \mathbb{R}^{n\times n}$ are the mass, damping and stiffness matrices respectively. In order to completely quantify the uncertainties associated with system (1) we need to obtain the probability density functions of the random matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. Using the parametric approach, such as the stochastic finite element method, one usually obtains a problem specific covariance structure for the elements of system matrices. The nonparametric approach [1–3, 7, 46–48] on the other hand results in a central Wishart distribution for the system matrices. In a recent paper [4] it was shown that a single Wishart matrix with properly selected parameters can be used for systems with both parametric uncertainty and nonparametric uncertainty. The calculation of the statistics of dynamic response can be expressed in terms of the eigenvalues and eigenvectors of a random matrix. This paper is focused on the density of eigenvalues of such random systems.

The outline of the paper is as follows. In section 2 dynamic response of linear stochastic systems is discussed. A brief overview of random matrix models in probabilistic structural dynamics is given in section 3. The density of the eigenvalues are discussed section 4. In section 5 the accuracy of the proposed results regarding the density of the eigenvalues are numerically verified. Based on the study taken in the paper, a set of conclusions are drawn in section 6.

2 Uncertainty quantification of dynamic response

Assuming all the initial conditions are zero and taking the Laplace transform of the equation of motion (1) we have

$$\left[s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right] \bar{\mathbf{q}}(s) = \bar{\mathbf{f}}(s) \tag{2}$$

where $\bar{(\bullet)}$ denotes the Laplace transform of the respective quantities. The aim here is to obtain the statistical properties of $\bar{\mathbf{q}}(s) \in \mathbb{C}^n$ when the system matrices are random matrices. The undamped eigenvalue problem is given by

$$\mathbf{K}\boldsymbol{\phi}_j = \omega_j^2\mathbf{M}\boldsymbol{\phi}_j, \quad j = 1, 2, \ldots, n \tag{3}$$

where ω_j^2 and $\boldsymbol{\phi}_j$ are respectively the eigenvalues and mass-normalized eigenvectors of the system. We define the matrices

$$\boldsymbol{\Omega} = \mathrm{diag}\,[\omega_1, \omega_2, \ldots, \omega_n] \in \mathbb{R}^{n\times n} \quad \text{and} \quad \boldsymbol{\Phi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n] \in \mathbb{R}^{n\times n} \tag{4}$$

so that

$$\boldsymbol{\Phi}^T\mathbf{K}_e\boldsymbol{\Phi} = \boldsymbol{\Omega}^2 \quad \text{and} \quad \boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \mathbf{I}_n \tag{5}$$

where $\mathbf{I}_n$ is an n-dimensional identity matrix. Using these, Eq. (2) can be transformed into the modal coordinates as

$$\left[s^2\mathbf{I}_n + s\mathbf{C}' + \boldsymbol{\Omega}^2\right] \bar{\mathbf{q}}' = \bar{\mathbf{f}}' \tag{6}$$

where and $(\bullet)'$ denotes the quantities in the modal coordinates:

$$\mathbf{C}' = \boldsymbol{\Phi}^T\mathbf{C}\boldsymbol{\Phi}, \quad \bar{\mathbf{q}} = \boldsymbol{\Phi}\bar{\mathbf{q}}' \quad \text{and} \quad \bar{\mathbf{f}}' = \boldsymbol{\Phi}^T\bar{\mathbf{f}} \tag{7}$$

For simplicity let us assume that the system is proportionally damped with deterministic modal damping factors $\zeta_1, \zeta_2, \ldots, \zeta_n$. Therefore, when we consider random systems, the matrix of eigenvalues $\boldsymbol{\Omega}^2$ in equation (6) will be a random matrix of dimension n. Suppose this random matrix is denoted by $\boldsymbol{\Xi} \in \mathbb{R}^{n\times n}$:

$$\boldsymbol{\Omega}^2 \sim \boldsymbol{\Xi} \tag{8}$$

Since $\boldsymbol{\Xi}$ is a symmetric and positive definite matrix, it can be diagonalized by a orthogonal matrix $\boldsymbol{\Psi}_r$ such that

$$\boldsymbol{\Psi}_r^T \boldsymbol{\Xi} \boldsymbol{\Psi}_r = \boldsymbol{\Omega}_r^2 \tag{9}$$

Here the subscript r denotes the random nature of the eigenvalues and eigenvectors of the random matrix $\boldsymbol{\Xi}$. Recalling that $\boldsymbol{\Psi}_r^T \boldsymbol{\Psi}_r = \mathbf{I}_n$, from equation (6) we obtain

$$\bar{\mathbf{q}}' = \left[s^2\mathbf{I}_n + s\mathbf{C}' + \boldsymbol{\Omega}^2\right]^{-1} \bar{\mathbf{f}}' \tag{10}$$

$$= \boldsymbol{\Psi}_r \left[s^2\mathbf{I}_n + 2s\boldsymbol{\zeta}\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_r^2\right]^{-1} \boldsymbol{\Psi}_r^T \bar{\mathbf{f}}' \tag{11}$$

where

$$\boldsymbol{\zeta} = \operatorname{diag}\left[\zeta_1, \zeta_2, \ldots, \zeta_n\right] \in \mathbb{R}^{n\times n} \tag{12}$$

The response in the original coordinate can be obtained as

$$\begin{aligned}\bar{\mathbf{q}}(s) = \boldsymbol{\Phi}\bar{\mathbf{q}}'(s) &= \boldsymbol{\Phi}\boldsymbol{\Psi}_r \left[s^2\mathbf{I}_n + 2s\boldsymbol{\zeta}\boldsymbol{\Omega}_r + \boldsymbol{\Omega}_r^2\right]^{-1} (\boldsymbol{\Phi}\boldsymbol{\Psi}_r)^T \bar{\mathbf{f}}(s) \\ &= \sum_{j=1}^{n} \frac{\mathbf{x}_{r_j}^T \bar{\mathbf{f}}(s)}{s^2 + 2s\zeta_j\omega_{r_j} + \omega_{r_j}^2} \mathbf{x}_{r_j}.\end{aligned} \tag{13}$$

Here

$$\boldsymbol{\Omega}_r = \operatorname{diag}\left[\omega_{r_1}, \omega_{r_2}, \ldots, \omega_{r_n}\right] \tag{14}$$

$$\text{and} \quad \mathbf{X}_r = \boldsymbol{\Phi}\boldsymbol{\Psi}_r = \left[\mathbf{x}_{r_1}, \mathbf{x}_{r_2}, \ldots, \mathbf{x}_{r_n}\right] \tag{15}$$

are respectively the matrices containing random eigenvalues and eigenvectors of the system. The Frequency Response Function (FRF) of the system can be obtained by substituting $s = \mathrm{i}\omega$ in Eq. (13). In the next section we discuss the derivation of the random matrix $\boldsymbol{\Xi}$.

3 Wishart random matrix model

We start with the fact that the baseline model of the system under consideration is known. Since proportional damping model is assumed, the baseline model consist of the mass and stiffness matrices given by $\mathbf{M}_0 \in \mathbb{R}^{n\times n}$ and $\mathbf{K}_0 \in \mathbb{R}^{n\times n}$. These matrices are in general large banded matrices and can be obtained using the conventional finite element method [10, 12, 13, 51]. In addition to this, it is assumed that the dispersion parameters associated with these matrices are known. The dispersion parameter [46, 47] is a measure of uncertainty in the system and it is similar to normalized variance of a matrix. For example, the dispersion parameter associated with the mass matrix is defined as

$$\delta_M = \frac{\mathrm{E}\left[\|\mathbf{M} - \mathbf{M}_0\|_{\mathrm{F}}^2\right]}{\|\mathbf{M}_0\|_{\mathrm{F}}^2} \tag{16}$$

where $\|\bullet\|_{\mathrm{F}}$ denotes the Frobenius norm of a matrix. The dispersion parameter associated with the stiffness matrix can be defined in a similar way. The dispersion parameters δ_M and δ_K can be obtained using stochastic finite element method [1–3] or experimental measurements [7].

From the mathematical analysis in the previous section it can be seen that the statistics of dynamic response depends only on the distribution of the eigenvalues and eigenvectors of the matrix $\boldsymbol{\Xi}$. Adhikari has shown that [4] the matrix $\boldsymbol{\Xi}$ can be modelled as a Wishart random matrix so that $\boldsymbol{\Xi} \sim W_n(p, \boldsymbol{\Sigma})$. We refer to the books by Muirhead [39], Eaton [14], Griko

[18], Gupta and Nagar [19], Mathai and Provost [33], Tulino and Verdú [50] and Mezzadri and Snaith [37] for discussions on Wishart random matrices and and related mathematical details. The parameters p and $\boldsymbol{\Sigma}$ can be obtained from the available data regarding the system, namely $\mathbf{M}_0$, $\mathbf{K}_0$, δ_M and δ_K. It was shown that

$$\boldsymbol{\Xi} \sim W_n\left(p, \boldsymbol{\Sigma}\right) \tag{17}$$

where

$$\boldsymbol{\Sigma} = \boldsymbol{\Omega}_0^2/\theta, \quad p = n + 1 + \theta \quad \text{and} \quad \theta = \frac{(1+\beta_H)}{\delta_H^2} - (n+1) \tag{18}$$

The constant β_H and the dispersion parameter δ_H can be obtained as

$$\beta_H = \left(\sum_{j=1}^{n} \omega_{0_j}^2\right)^2 / \sum_{j=1}^{n} \omega_{0_j}^4 \tag{19}$$

and

$$\begin{aligned}
\delta_H = & \frac{\left(\left(\beta n^2 + 2\beta n + n^2 - 1 + \beta\right)\delta_K - n\beta_K - \beta n\beta_K - \beta n - n - \beta\beta_K + 1 - \beta + \beta_K\right){\delta_M}^2}{\left(1+\beta_K\right)\left(-1-\beta_M+n\delta_M\right)\left(-1-\beta_M+n\delta_M+3\delta_M\right)} \\
& + \frac{\left(-2n - 2n\beta_M - 2\beta - 2\beta\beta_1 - 2\beta n - 2\beta n\beta_M\right)\delta_K\delta_M}{\left(1+\beta_K\right)\left(-1-\beta_M+n\delta_M\right)\left(-1-\beta_M+n\delta_M+3\delta_M\right)} \\
& + \frac{\left(\beta\beta_K\beta_M + \beta_K\beta_M + \beta\beta_M + \beta_K + \beta\beta_K + \beta + 1 + \beta_M\right)\delta_M}{\left(1+\beta_K\right)\left(-1-\beta_M+n\delta_M\right)\left(-1-\beta_M+n\delta_M+3\delta_M\right)} \\
& + \frac{\left(\beta{\beta_M}^2 + 2\beta\beta_M + 2\beta_M + \beta + 1 + {\beta_M}^2\right)\delta_K}{\left(1+\beta_K\right)\left(-1-\beta_M+n\delta_M\right)\left(-1-\beta_M+n\delta_M+3\delta_M\right)}
\end{aligned} \tag{20}$$

where

$$\beta_M = \left\{\text{Trace}\left(\mathbf{M}_0\right)\right\}^2 / \text{Trace}\left(\mathbf{M}_0{}^2\right) \quad \text{and} \quad \beta_K = \left\{\text{Trace}\left(\mathbf{K}_0\right)\right\}^2 / \text{Trace}\left(\mathbf{K}_0{}^2\right) \tag{21}$$

These relationships completely defines all the parameters of the Wishart random matric necessary for uncertainty quantification of structural dynamic systems.

4 Density of eigenvalues

From equations (10) and (11) it is clear that the spectral properties of the Wishart random matrix $\boldsymbol{\Xi}$ play a key role in uncertainty quantification of stochastic dynamical systems. In this section we specifically look into the density of the eigenvalues. Our main result is that the density of the eigenvalues have the 'self averaging' property. This implies that the density of the eigenvalues of nominally identical systems are almost identical. In the next two subsections we aim to present a rigorous proof.

4.1 Linear eigenvalue statistic

Let $\boldsymbol{\Xi}$ be a $n \times n$ random matrix and $\{\lambda_l\}_{l=1}^n$ its eigenvalues. Then the (empirical) eigenvalue density is

$$\rho_n(\lambda) = n^{-1} \sum_{l=1}^{n} \delta(\lambda - \lambda_l), \tag{22}$$

where δ is the Dirac delta-function. Denote $\overline{\rho}_n(\lambda)$ the expectation of ρ_n, i.e.,

$$\overline{\rho}_n(\lambda) = \mathbf{E}\{\rho_n(\lambda)\}. \tag{23}$$

The symbol $\mathbf{E}\{...\}$ denotes the mathematical expectation, i.e., the averaging with respect to the corresponding probability law.

To prove the selfaveraging of ρ_n one has to show that, say, its the variance

$$\mathbf{Var}\{\rho_n(\lambda)\} := \mathbf{E}\{\rho_n^2(\lambda)\} - \mathbf{E}^2\{\rho_n(\lambda)\} \tag{24}$$

tends to zero as $n \to \infty$, and if possible, to find (or to estimate) the rate of decay of the variance. However, while $\overline{\rho}_n(\lambda) = \mathbf{E}\{\rho_n(\lambda)\}$, i.e., the second term above, is well defined, this is not the case for the first term. Indeed, we have by definition

$$\rho_n^2(\lambda) = n^{-2} \sum_{l_1,l_2=1}^{n} \delta(\lambda - \lambda_{l_1})\delta(\lambda - \lambda_{l_2}) \tag{25}$$

$$= n^{-2} \sum_{l_1=1}^{n} \delta^2(\lambda - \lambda_{l_1}) + n^{-2} \sum_{l_1 \neq l_2} \delta(\lambda - \lambda_{l_1})\delta(\lambda - \lambda_{l_2}). \tag{26}$$

and we see that summand $\delta^2(\lambda - \lambda_{l_1})$ of the first sum on the r.h.s is not well defined (it is often said that the square of delta-function is infinity).

To avoid this we have to 'smooth' the delta-function, i.e., to replacing it with a smooth function having a well pronounced peak. If we denote this function u, then we have

$$n^{-1} \sum_{l=1}^{n} u(\lambda - \lambda_l) = \int u(\lambda - \mu)\rho_n(\mu)d\mu \tag{27}$$

instead of $\rho_n(\lambda)$. This happens, in particular, when one computes $\rho_n(\lambda)$ numerically. This is because one first finds the eigenvalues and then draws a continuous envelope curve which corresponds to smoothing ρ_n with a function u whose peak has a width bigger that the distance (of the order $O(n^{-1})$) between the eigenvalues. This is why we will not deal with $\rho_n(\lambda)$ itself but rather with so called *linear eigenvalue statistics*, defined for any sufficiently smooth *test function* φ as

$$N_n[\varphi] = n^{-1} \sum_{l=1}^{n} \varphi(\lambda_l) = \int \varphi(\mu)\rho_n(\mu)d\mu \tag{28}$$

Note that ρ_n in equation (22) correspond formally to $\varphi(\mu) = \delta(\lambda - \mu)$ for a given λ. In the next subsection, we consider the density of eigenvalues within these general frameworks of the Random Matrix Theory.

4.2 Self averaging property and the Marčenko-Pastur density

Let $\boldsymbol{\Xi}$ be $n \times n$ real symmetric or Hermitian Wishart matrix with p degrees of freedom and a $n \times n$ covariance matrix $\boldsymbol{\Sigma}$, i.e., $\boldsymbol{\Xi} \sim W_n(p, \boldsymbol{\Sigma})$. Denote $\{\lambda_l\}_{l=1}^n$ its eigenvalues and consider the *linear eigenvalue statistic* (see (28))

$$N_n[\varphi] = n^{-1} \sum_{l=1}^{n} \varphi(\lambda_l), \tag{29}$$

corresponding to a real or complex valued *test function* φ. It can be shown (see [17, 32, 45]) that

$$\lim_{n\to\infty\to,\ p\to\infty,\ p/n\to c\in[1,\infty)} \mathbf{E}\{N_n[\varphi]\} = \int \varphi(\lambda)\rho(\lambda)d\lambda, \tag{30}$$

where ρ can be found by solving a certain functional equation for its Stieltjes transform

$$f(z) = \int \frac{\rho(\lambda)d\lambda}{\lambda - z},\ \Im z \neq 0. \tag{31}$$

In the case, where $\boldsymbol{\Sigma} = \mathbf{I}_n$ and $c > 1$ we have

$$\rho(\lambda) = \frac{1}{2\pi\lambda} \begin{cases} \sqrt{(a_+ - \lambda)(\lambda - a_-)}, & \lambda \in [a_-, a_+], \\ 0, & \lambda \notin [a_-, a_+], \end{cases} \tag{32}$$

where $a_\pm = (1 \pm \sqrt{c})^2$. If $c = 1$ then

$$\rho(\lambda) = \frac{1}{2\pi} \begin{cases} \sqrt{(4 - \lambda)/\lambda}, & \lambda \in (0, 4], \\ 0, & \lambda \notin (0, 4]. \end{cases} \tag{33}$$

The density of eigenvalues given by equations (32) or (33) is now known as Marčenko-Pastur (MP) density. This density will be considered in the next section in the numerical examples.

Let us show that the fluctuations of $N_n[\varphi]$ around its expectation $\mathbf{E}\{N_n[\varphi]\}$ vanish sufficiently fast in the limit

$$n \to \infty \to,\ p \to \infty,\ p/n \to\ c \in (0, \infty) \tag{34}$$

To this end we obtain a bound for the variance

$$\mathbf{Var}\{N_n[\varphi]\} = \mathbf{E}\{|N_n[\varphi]|^2\} - |\mathbf{E}\{N_n[\varphi]\}^2$$

of $N_n[\varphi]$. The bound is

$$\mathbf{Var}\{N_n[\varphi]\} \le \frac{4\sqrt{3}}{n^2 p} \mathrm{Tr}\ \boldsymbol{\Sigma}^2 (\max_{\lambda \in \mathbb{R}} |\varphi'(\lambda)|)^2. \tag{35}$$

It is valid for real symmetric as well as for hermitian Wishart matrices. We give below its proof for real symmetric matrices. The proof for hermitian matrices is practically the same.

It can be shown that if we want to keep the spectrum of $\boldsymbol{\Xi}$ bounded for all n, p of (34) rather than escaping to infinity, we have to assume that in the limit (34):

$$\max_{p,n} n^{-1} \mathrm{Tr} \boldsymbol{\Sigma}^2 \le C < \infty. \tag{36}$$

(the same, in fact stronger, condition is necessary to prove (30)). Assuming this and

$$\max_{\lambda \in \mathbb{R}} |\varphi'(\lambda)| < \infty, \tag{37}$$

we obtain from (35) that

$$\mathbf{Var}\{N_n[\varphi]\} = O(n^{-2}) \tag{38}$$

under condition (34) and (36) – (37).

Note that if $\{\lambda_l\}_{l=1}^n$ were independent identically distributed random variables, then the variance of their linear statistics is equal to $n^{-1}\mathbf{Var}\{\varphi(\lambda_1)\}$, i.e., is $O(n^{-1})$ for any φ such that $\mathbf{Var}\{\varphi(\lambda_1)\} < \infty$. This is a manifestation of strong statistical dependence between eigenvalues of Wishart (and many other) random matrices, known also as the repulsion of eigenvalues and/or the rigidity of spectrum (see e.g. [36]).

Proof of (35). The proof can be given by following these steps:

(i). Given the standard Gaussian random variables $\{\xi_l\}_{l=1}^q$

$$\mathbf{E}\{\xi_l\} = 0,\ \mathbf{E}\{\xi_l \xi_m\} = \delta_{lm} \tag{39}$$

and a differentiable function $\Phi : \mathbb{R}^q \to \mathbb{C}$ of q variables, consider the random variable

$$\Psi = \Phi(\xi_1, ..., \xi_q). \tag{40}$$

Then its variance admits the bound

$$\mathbf{Var}\{\Psi\} \leq \sum_{l=1}^{q} \mathbf{E}\left\{\left|\frac{\partial \Phi}{\partial \xi_l}\right|^2\right\}, \tag{41}$$

known as the Poincaré inequality (see e.g. [11]).

(ii). Given a $n \times n$ real symmetric or hermitian matrix $\mathbf{A}(t)$ depending on a parameter t and a function $\varphi : \mathbb{R} \to \mathbb{C}$, consider the matrix function $\varphi(\mathbf{A}(t))$. Then we have

$$\frac{d}{dt}\mathrm{Tr}\varphi(\mathbf{A}(t)) = \mathrm{Tr}\varphi'(\mathbf{A}(t))\mathbf{A}'(t)). \tag{42}$$

Note now that we can write the $W_n(p, \boldsymbol{\Sigma})$ real symmetric Wishart matrix as

$$\boldsymbol{\Xi} = p^{-1}\mathbf{R}\mathbf{X}\mathbf{X}^T\mathbf{R}, \tag{43}$$

where $\mathbf{R}$ is a positive definite $n \times n$ matrix such that $\mathbf{R}^2 = \boldsymbol{\Sigma}$ and $\mathbf{X} = \{\mathbf{X}_{\alpha j}\}_{\alpha,j=1}^{p,n}$ is a $p \times n$ random matrix whose entries are the standard Gaussian random variables

$$\mathbf{E}\{\mathbf{X}_{\alpha j}\} = 0,\ \mathbf{E}\{\mathbf{X}_{\alpha j}\mathbf{X}^*_{\beta k}\} = \delta_{\alpha\beta}\delta_{jk}. \tag{44}$$

By using this one can check easily that the entries $\{\boldsymbol{\Xi}_{jk}\}_{j,k=1}^{n}$ of $\boldsymbol{\Xi}$ are

$$\boldsymbol{\Xi}_{jk} = p^{-1}\sum_{j,k=1}^{n}\sum_{\alpha=1}^{p}\mathbf{R}_{jl}\mathbf{X}_{\alpha l}\mathbf{X}_{\alpha m}\mathbf{R}_{mk} \tag{45}$$

thus

$$\mathbf{E}\{\boldsymbol{\Xi}_{jk}\} = \boldsymbol{\Sigma}_{jk} \tag{46}$$

as it should be.

Note now that it follows from the spectral theorem for real symmetric matrices and (29) that

$$N_n[\varphi] = \mathrm{Tr}\varphi(\boldsymbol{\Xi}). \tag{47}$$

Take in (41) $n^{-1}\mathrm{Tr}\varphi(\boldsymbol{\Xi})$ as Ψ and $\{\mathbf{X}_{\alpha j}\}_{\alpha,j=1}^{p,n}$ as $\{\xi_l\}_{l=1}^{q}$, hence $q = np$. This yields

$$\mathbf{Var}\{N_n[\varphi]\} \leq n^{-2}\sum_{\alpha=1}^{p}\sum_{j=1}^{n}\mathbf{E}\left\{\left|\frac{\partial \mathrm{Tr}\varphi(\boldsymbol{\Xi})}{\partial \mathbf{X}_{\alpha j}}\right|^2\right\}. \tag{48}$$

Take now in (42) $\mathbf{X}_{\alpha j}$ as t, $p^{-1}RXX^T\mathbf{R}$ as $\mathbf{A}$ and use the formula (see (45))

$$\frac{\partial}{\partial \mathbf{X}_{\alpha j}}(p^{-1}\mathbf{R}\mathbf{X}\mathbf{X}^T\mathbf{R})_{lm} = \mathbf{R}_{lj}(\mathbf{X}^T\mathbf{R})_{\alpha m} + (\mathbf{R}\mathbf{X})_{l\alpha}\mathbf{R}_{jm}. \tag{49}$$

This yields after a simple algebra

$$\mathbf{Var}\{N_n[\varphi]\} \leq \frac{4}{(np)^2}\mathbf{E}\left\{\mathrm{Tr}\boldsymbol{\Xi}\varphi'(\boldsymbol{\Xi})\boldsymbol{\Sigma}\overline{\varphi'}(\boldsymbol{\Xi})\right\}. \tag{50}$$

Now we use

(iii). The Schwarz inequality for traces

$$|\mathrm{Tr}\mathbf{A}\mathbf{B}|^2 \leq \mathrm{Tr}\mathbf{A}\mathbf{A}^*\ \mathrm{Tr}\mathbf{B}\mathbf{B}^*$$

with $\mathbf{A} = \boldsymbol{\Xi}\varphi'$ and $\mathbf{B} = \boldsymbol{\Sigma}\varphi'$. We obtain that

$$|\mathrm{Tr}\boldsymbol{\Xi}\varphi'(\boldsymbol{\Xi})\boldsymbol{\Sigma}\overline{\varphi'}(\boldsymbol{\Xi})| \leq (\mathrm{Tr}\boldsymbol{\Xi}^2\varphi'(\boldsymbol{\Xi})\overline{\varphi'}(\boldsymbol{\Xi}))^{1/2}(\mathrm{Tr}\boldsymbol{\Sigma}^2\varphi'(\boldsymbol{\Xi})\overline{\varphi'}(\boldsymbol{\Xi}))^{1/2}.$$

Two more inequality to use are

(iv) $|\mathrm{Tr}\mathbf{AB}| \leq ||\mathbf{A}||\mathrm{Tr}\mathbf{B}$,

valid for any matrix $\mathbf{A}$ and a positive definite $\mathbf{B}$, where $||\mathbf{A}||$ is the Euclidian norm of $\mathbf{A}$, and

(v) $||\psi(\boldsymbol{\Xi})|| \leq \max_{x\in\mathbb{R}} |\psi(x)|$,

valid a real symmetric (hermitian) $\boldsymbol{\Xi}$.

We obtain from the above

$$\mathbf{Var}\{N_n[\varphi]\} \leq \frac{4}{(np)^2}(\max_{x\in\mathbb{R}} |\varphi'(x)|)^2(\mathrm{Tr}\boldsymbol{\Sigma}^2)^{1/2}(\mathbf{E}\{\mathrm{Tr}\boldsymbol{\Xi}^2\})^{1/2} \tag{51}$$

It follows from (44) and (45) that if $n \leq p$, then

$$\mathbf{E}\{\mathrm{Tr}\boldsymbol{\Xi}^2\} \leq 3p^2\mathrm{Tr}\boldsymbol{\Sigma}^2.$$

Plugging this in (51), we obtain (35). In the next section, the validity of equations and (38) are examined using numerical examples.

5 Numerical investigations

In the previous section it was proved that for large random dynamical systems, the density of eigenvalues reaches a non-random limit. In this section we examine the validity of this result using numerical examples. We also verify if one of most widely used asymptotic density, namely the Marčenko-Pastur density, is valid for structural-dynamic systems.

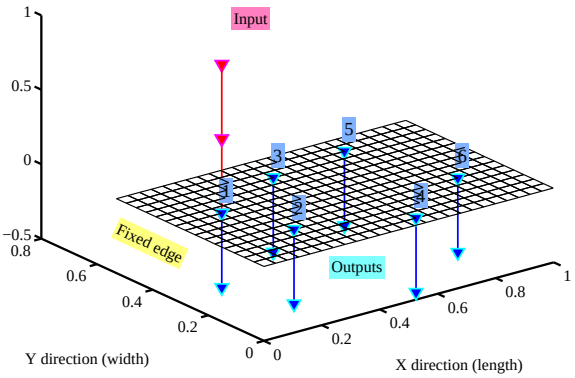

Fig. 1 The Finite Element (FE) model of a steel cantilever plate. The deterministic properties are: $\bar{E} = 200 \times 10^9 \mathrm{N/m^2}$, $\bar{\mu} = 0.3$, $\bar{\rho} = 7860\mathrm{kg/m^3}$, $\bar{t} = 3.0\mathrm{mm}$, $L_x = 0.998\mathrm{m}$, $L_y = 0.59\mathrm{m}$.

A rectangular cantilever steel plate is considered to illustrate the convergence of the eigenvalue-density. The deterministic properties are assumed to be $\bar{E} = 200\times10^9\mathrm{N/m^2}$, $\bar{\mu} = 0.3$, $\bar{\rho} = 7860\mathrm{kg/m^3}$, $\bar{t} = 3.0\mathrm{mm}$, $L_x = 0.998\mathrm{m}$, $L_y = 0.59\mathrm{m}$. The schematic diagram of the plate is shown in Figure 1. The plate is divided into 25 elements along the x-axis and 15 elements along the y-axis for the numerical calculations. The resulting system has 1200 degrees of freedom so that $n = 1200$. In Figure 2 the density of eigenvalues of the deterministic system is compared with the Marčenko-Pastur density. Except in the very low frequency region, the Marčenko-Pastur density agree well with the Finite Element results. Two different cases of uncertainties are considered. In the first case it is assumed that the material properties are randomly inhomogeneous. In the second case we consider that the plate is 'perturbed' by attaching spring-mass oscillators at random locations. The first case corresponds to a parametric uncertainty problem while the second case corresponds to a non-parametric uncertainty problem.

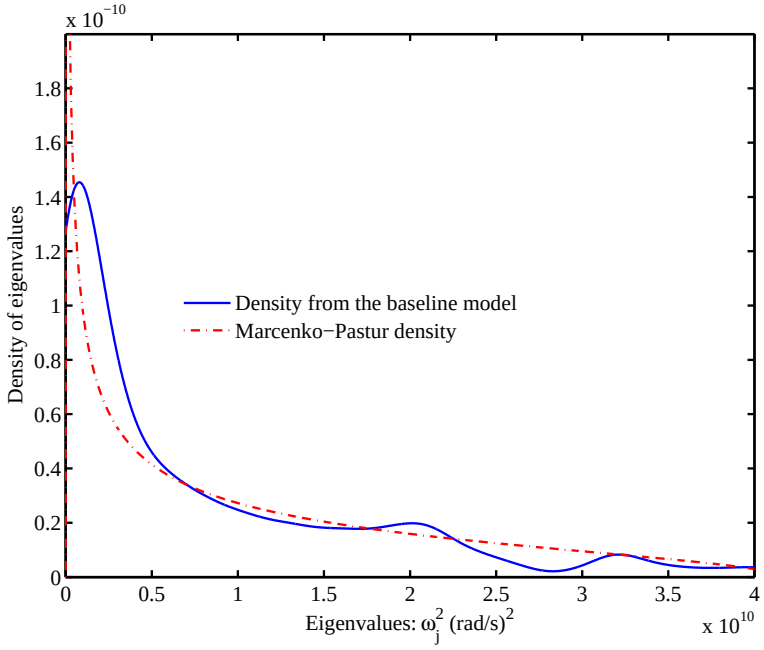

Fig. 2 The density of 1200 eigenvalues of the baseline model.

5.1 Plate with randomly inhomogeneous material properties: parametric uncertainty problem

It is assumed that the Young's modulus, Poissons ratio, mass density and thickness are random fields of the form

$$E(\mathbf{x}) = \bar{E}\left(1 + \epsilon_E f_1(\mathbf{x})\right), \quad \mu(\mathbf{x}) = \bar{\mu}\left(1 + \epsilon_\mu f_2(\mathbf{x})\right) \tag{52}$$

$$\rho(\mathbf{x}) = \bar{\rho}\left(1 + \epsilon_\rho f_3(\mathbf{x})\right) \quad \text{and} \quad t(\mathbf{x}) = \bar{t}\left(1 + \epsilon_t f_4(\mathbf{x})\right) \tag{53}$$

The two dimensional vector $\mathbf{x}$ denotes the spatial coordinates. The strength parameters are assumed to be $\epsilon_E = 0.10$, $\epsilon_\mu = 0.10$, $\epsilon_\rho = 0.08$ and $\epsilon_t = 0.12$. The random fields $f_i(\mathbf{x}), i = 1, \cdots, 4$ are assumed to be correlated homogenous Gaussian random fields. An exponential correlation function with correlation length 0.2 times the lengths in each direction has been considered. The random fields are simulated by expanding them using the Karhunen-Loève expansion [16, 41] involving uncorrelated standard normal variables. A 5000-sample Monte Carlo simulation is performed to obtain the eigenvalues of the system. In Figure 3 100 samples of the density of the eigenvalues are shown, alongside the fitted Marčenko-Pastur density and density obtained from the baseline model. The density of the eigenvalues of the random realization are quite close.

5.2 Plate with randomly attached spring-mass oscillators: nonparametric uncertainty problem

In this example we consider the same plate but with non-parametric uncertainty. The baseline model is perturbed by attaching 10 spring mass oscillators with random natural frequencies at random nodal points in the plate. The natural frequencies of the attached oscillators follow a uniform distribution between 0.2 kHz to 4.0 kHz. The nature of uncertainty in this case is different from the previous case because here the sparsity structure of the system matrices change with different realizations of the system. Again a 5000-sample Monte Carlo simulation is performed to obtain the eigenvalues. In Figure 4, 100 samples of the density of the eigenvalues are shown, alongside the fitted Marčenko-Pastur density and density obtained from the baseline model. The density of the eigenvalues of the random realization are quite close.

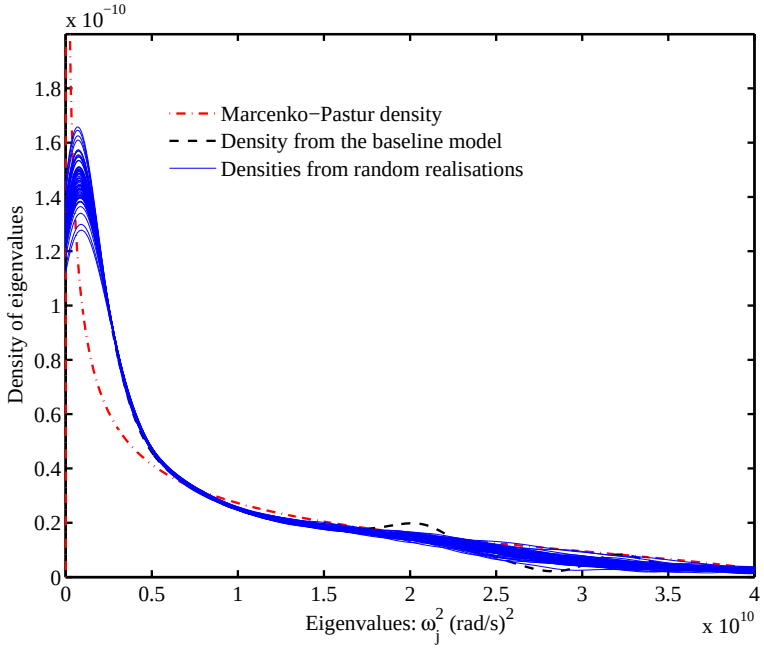

Fig. 3 The density of eigenvalues of the plate with randomly inhomogeneous material properties.

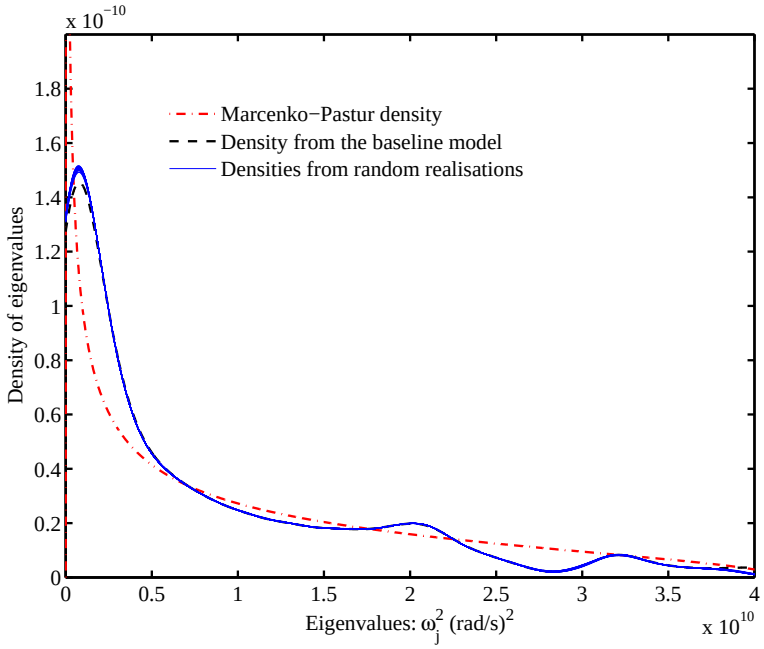

Fig. 4 The density of eigenvalues of the plate with randomly attached oscillators.

6 Conclusions

The density of eigenvalues of structural dynamical systems with uncertainty is considered in this paper. Due to the positive definiteness nature of a real system, it can be modeled using a Wishart random matrix with suitable parameters. These parameters in turn can be explicitly obtained form the baseline model and dispersion parameters corresponding to the mass and stiffness matrices of the system using the closed-form expressions given the paper. It was shown that for large random systems, the density of eigenvalues reaches a non-random limit. It

particular, it was rigorously proved that for an n-dimensional system, the variance associated with a suitable linear statistic of the eigenvalues is in the order $O(n^{-2})$. This result shows that if a system is large, then the detailed nature of random perturbation do not effect the eigenvalue-density. Under certain simplified assumptions, this asymptotic density can be suitably represented by the so called Marčenko-Pastur density. Two numerical examples involving a cantilever plate with parametric and non-parametric uncertainty have been used to investigate the validity of analytical results. Using direct Monte Carlo simulations, it was indeed observed that eigenvalue-densities of nominally identical systems do not differ from each other and are very close to the Marčenko-Pastur density. It is important to note that the original matrices are not Wishart matrices, but the eigenvalue-density is close to that of Wishart matrices.

The strong convergence of the eigenvalue-density perhaps explains why many random matrix based methods (e.g. statistical energy analysis) are so useful for high-frequency vibration problems where the sizes of the underlying matrices are very large. This convergence property may also opens up the possibility of calculating other useful quantities such as the response statistics.

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